Searching for Universal Truths
Abstract Algebra

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Navigating Mathematical and Statistical Territories

Notations

- sets of numbers
  - $\mathbf{N}$ - set of natural numbers
  - $\mathbf{Z}$ - set of integers
  - $\mathbf{Z}_+$ - set of nonnegative integers
  - $\mathbf{Q}$ - set of rational numbers
  - $\mathbf{R}$ - set of real numbers
  - $\mathbf{R}_+$ - set of nonnegative real numbers
  - $\mathbf{R}_{++}$ - set of positive real numbers
  - $\mathbf{C}$ - set of complex numbers
- sequences $\langle x_i \rangle$ and the like
  - finite $\langle x_i \rangle_{i=1}^{n}$, infinite $\langle x_i \rangle_{i=1}^{\infty}$ - use $\langle x_i \rangle$ whenever unambiguously understood
  - similarly for other operations, *e.g.*, $\sum x_i$, $\prod x_i$, $\cup A_i$, $\cap A_i$, $\bigtimes A_i$
  - similarly for integrals, *e.g.*, $\int f$ for $\int_{-\infty}^{\infty} f$
- sets
  - $\tilde{A}$ - complement of A

- $A \sim B$ - $A \cap \tilde{B}$
- $A \Delta B$ - $(A \cap \tilde{B}) \cup (\tilde{A} \cap B)$
- $\mathcal{P}(A)$ - set of all subsets of A

- sets in metric vector spaces
  - $\overline{A}$ - closure of set A
  - A° - interior of set A
  - $\mathbf{relint}\, A$ - relative interior of set A
  - $\mathbf{bd}\, A$ - boundary of set A

- set algebra
  - $\sigma(\mathcal{A})$ - σ-algebra generated by $\mathcal{A}$, *i.e.*, smallest σ-algebra containing $\mathcal{A}$

- norms in $\mathbf{R}^n$
  - $\|x\|_p$ $(p \geq 1)$ - p-norm of $x \in \mathbf{R}^n$, *i.e.*, $(|x_1|^p + \cdots + |x_n|^p)^{1/p}$
  - *e.g.*, $\|x\|_2$ - Euclidean norm

- matrices and vectors
  - a_i - i-th entry of vector a
  - A_{ij} - entry of matrix A at position (i, j), *i.e.*, entry in i-th row and j-th column
  - $\mathbf{Tr}(A)$ - trace of $A \in \mathbf{R}^{n \times n}$, *i.e.*, $A_{1,1} + \cdots + A_{n,n}$

- symmetric, positive definite, and positive semi-definite matrices
  - $\mathbf{S}^n \subset \mathbf{R}^{n\times n}$ - set of symmetric matrices
  - $\mathbf{S}^n_+ \subset \mathbf{S}^n$ - set of positive semi-definite matrices; $A \succeq 0 \Leftrightarrow A \in \mathbf{S}^n_+$
  - $\mathbf{S}^n_{++} \subset \mathbf{S}^n$ - set of positive definite matrices; $A \succ 0 \Leftrightarrow A \in \mathbf{S}^n_{++}$
- sometimes, use Python script-like notations (with serious abuse of mathematical notations)
  - use $f : \mathbf{R} \to \mathbf{R}$ as if it were $f : \mathbf{R}^n \to \mathbf{R}^n$, *e.g.*,

$$\exp(x) = (\exp(x_1), \ldots, \exp(x_n)) \quad \text{for } x \in \mathbf{R}^n$$

    and

$$\log(x) = (\log(x_1), \ldots, \log(x_n)) \quad \text{for } x \in \mathbf{R}^n_{++}$$

    which corresponds to Python code `numpy.exp(x)` or `numpy.log(x)` where `x` is instance of `numpy.ndarray`, *i.e.*, `numpy` array
  - use $\sum x$ to mean $\mathbf{1}^T x$ for $x \in \mathbf{R}^n$, *i.e.*

$$\sum x = x_1 + \cdots + x_n$$

    which corresponds to Python code `x.sum()` where `x` is `numpy` array

– use x/y for $x, y \in \mathbf{R}^n$ to mean

$$\left[\begin{array}{ccc} x_1/y_1 & \cdots & x_n/y_n \end{array}\right]^T$$

which corresponds to Python code `x / y` where `x` and `y` are 1-d `numpy` arrays

– use X/Y for $X, Y \in \mathbf{R}^{m\times n}$ to mean

$$\left[\begin{array}{cccc} X_{1,1}/Y_{1,1} & X_{1,2}/Y_{1,2} & \cdots & X_{1,n}/Y_{1,n} \\ X_{2,1}/Y_{2,1} & X_{2,2}/Y_{2,2} & \cdots & X_{2,n}/Y_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m,1}/Y_{m,1} & X_{m,2}/Y_{m,2} & \cdots & X_{m,n}/Y_{m,n} \end{array}\right]$$

which corresponds to Python code `X / Y` where `X` and `Y` are 2-d `numpy` arrays

Some definitions

Definition 1. [infinitely often - i.o.] *statement P_n, said to happen* infinitely often *or* i.o. *if*

$$(\forall N \in \mathbf{N})\,(\exists n > N)\,(P_n)$$

Definition 2. [almost everywhere - a.e.] *statement $P(x)$, said to happen* almost everywhere *or* a.e. *or* almost surely *or* a.s. *(depending on context) associated with measure space $(X, \mathscr{B}, \mu)$ if*

$$\mu\{x|P(x)\} = 1$$

or equivalently

$$\mu\{x| \sim P(x)\} = 0$$

Some conventions

- (for some subjects) use following conventions
  - $0 \cdot \infty = \infty \cdot 0 = 0$
  - $(\forall x \in \mathbf{R}_{++})(x \cdot \infty = \infty \cdot x = \infty)$
  - $\infty \cdot \infty = \infty$

Abstract Algebra

Why Abstract Algebra?

Why abstract algebra?

- it's fun!
- can understand *instrict structures* of algebraic objects
- allow us to solve *extremely practical problems* (depending on your definition of practicality)
  - $e.g.$, can prove why root formulas for polynomials of order $n \geq 5$ do not exist
- prepare us for pursuing further math topics such as
  - differential geometry
  - algebraic geometry
  - analysis
  - representation theory
  - algebraic number theory

Some history

- by the way, historically, often the case that application of an idea presented before extracting and presenting the idea on its own right

- $e.g.$, Galois used "quotient group" only implicitly in his 1830's investigation, and it had to wait until 1889 to be explicitly presented as "abstract quotient group" by Hölder

Groups

Monoids

Definition 3. [law of composition] *mapping $S \times S \to S$ for set S, called* law of composition (of S to itself)

- *when* $(\forall x, y, z \in S)((xy)z = x(yz))$, *composition is said to be* associative
- $e \in S$ *such that* $(\forall x \in S)(ex = xe = x)$, *called* unit element - *always unique*

  Proof: for any two unit elements e *and* f, $e = ef = f$, *hence,* $e = f$

Definition 4. [monoids] *set M with composition which is associative and having unit element, called* monoid *(so in particular, M is not empty)*

- *monoid* M *with* $(\forall x, y \in M)\,(xy = yx)$, *called* commutative or abelian *monoid*
- *subset* $H \subset M$ *which has* the *unit element* e *and is itself monoid, called* submonoid

Groups

Definition 5. [group] *monoid G with*

$$(\forall x \in G)\,(\exists y \in G)\,(xy = yx = e)$$

called group

- *for $x \in G$, $y \in G$ with $xy = yx = e$, called* inverse of x
- *group derived from commutative monoid, called* abelian group *or* commutative group
- *group G with $|G| < \infty$, called* finite group
- *(similarly as submonoid) $H \subset G$ that has unit element and is itself group, called* subgroup
- *subgroup consisting only of unit element, called* trivial

Cyclic groups, generators, and direct products

Definition 6. [cyclic groups] *group G with*

$$(\exists a \in G)\,(\forall x \in G)\,(\exists n \in \mathbf{N})\,(x = a^n)$$

called cyclic group, *such $a \in G$ called* cyclic generator

Definition 7. [generators] *for group G, $S \subset G$ with*

$$(\forall x \in G)\,(x \text{ is arbitrary product of elements or inverse elements of } S)$$

called set of generators for G, *said to* generate G, *denoted by $G = \langle S \rangle$*

Definition 8. [direct products] *for two groups G_1 and G_2, group $G_1 \times G_2$ with*

$$(\forall (x_1, x_2), (y_1, y_2) \in G_1 \times G_2)\,((x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2, y_2) \in G_1 \times G_2)$$

whose unit element defined by (e_1, e_2) where e_1 and e_2 are unit elements of G_1 and G_2 respectively, called direct product of G_1 and G_2

Homeomorphism and isomorphism

Definition 9. [homeomorphism] *for monoids M and M', mapping $f : M \to M'$ with $f(e) = e'$*

$$(x, y \in M) \, (f(xy) = f(x)f(y))$$

where e and e' are unit elements of M and M' respectively, called monoid-homeomorphism *or simple* homeomorphism

- group homeomorphism $f : G \to G'$ *is similarly monoid-homeomorphism*
- *homeomorphism $f : G \to G'$ where exists $g : G \to G'$ such that $f \circ g : G' \to G'$ and $g \circ f : G \to G$ are identity mappings, called* isomorphism, *sometimes denoted by* $G \approx G'$
- *homeomorphism of G into itself, called* endomorphism
- *isomorphism of G onto itself, called* automorphism

- set of all automorphisms of G is itself group, denoted by $\mathrm{Aut}(G)$

Kernel, image, and embedding of homeomorphism

Definition 10. [kernel of homeomorphism] *for group-homeomorphism $f : G \to G'$ where e' is unit element of G', $f^{-1}(\{e'\})$, which is subgroup of G, called* kernel of f, *denoted by* $\mathrm{Ker}\, f$

Definition 11. [embedding of homeomorphism] *homeomorphism $f : G \to G'$ establishing isomorphism between G and $f(G) \subset G'$, called* embedding

Proposition 1. [group homeomorphism and isomorphism]

- *for group-homeomorphism $f : G \to G'$, $f(G) \subset G'$ is subgroup of G'*
- *homeomorphism whose kernel is trivial is injective, often denoted by special arrow*

$$f : G \hookrightarrow G'$$

- *surjective homeomorphism whose kernel is trivial is* isomorphism
- *for group G, its generators S, and another group G', map $f : S \to G'$ has at most one extension to homeomorphism of G into G'*

Orthogonal subgroups

Proposition 2. [orthogonal subgroups] *for group G and two subgroups H and $K \subset G$ with $HK = G$, $H \cap K = \{e\}$, and $(x \in H, y \in K)\,(xy = yx)$,*

$$f : H \times K \to G$$

with $(x, y) \mapsto xy$ is isomorphism

can generalize to finite number of subgroups, $H_1, \ldots, H_n$ such that

$$H_1 \cdots H_n = G$$

and

$$H_{k+1} \cap (H_1 \cdots H_k) = \{e\}$$

in which case, G is isomorphic to $H_1 \cdots H_n$

Cosets of groups

Definition 12. [cosets of groups] *for group G and subgroup $H \subset G$, aH for some $a \in G$, called* left coset of H in G*, and element in aH, called* coset representation of aH *- can define* right cosets *similarly*

Proposition 3. [cosets of groups] *for group G and subgroup $H \subset G$,*

- *for $a \in G$, $x \mapsto ax$ induces bijection of H onto aH, hence all left cosets have same cardinality*
- *$aH \cap bH \neq \emptyset$ for $a, b \in G$ implies $aH = bH$*
- *hence, G is disjoint union of left cosets of H*
- *same statements can be made for right cosets*

Definition 13. [index and order of group] *number of left cosets of H in G, called* index of H in G*, denoted by $(G : H)$ - index of trivial subgroups, called* order of G*, denoted by $(G : 1)$*

Indices and orders of groups

Proposition 4. [indices and orders] *for group G and two subgroups H and $K \subset G$ with $K \subset H$,*

$$(G : H)(H : K) = (G : K)$$

when K is trivial, we have

$$(G : H)(H : 1) = (G : 1)$$

(proof can be found in Proof 1)

hence, if $(G : 1) < \infty$, both $(G : H)$ and $(H : 1)$ divide $(G : 1)$

Normal subgroup

Definition 14. [normal subgroups] *subgroup $H \subset G$ of group G with*

$$(\forall x \in G)\,(xH = Hx) \Leftrightarrow (\forall x \in G)\left(xHx^{-1} = H\right)$$

called normal subgroup of G, *in which case*

- *set of cosets $\{xH|x \in G\}$ with law of composition defined by $(xH)(yH) = (xy)H$, forms group with unit element H, denoted by G/H, called* factor group of G by H, *read* G modulo H *or* G mod H
- *$x \mapsto xH$ induces homeomorphism of X onto $\{xH|x \in G\}$, called* canonical map, *kernel of which is H*

Proposition 5. [normal subgroups and factor groups]

- *kernel of (every) homeomorphism of G is normal subgroups of G*
- *for family of normal subgroups of G, $\langle N_\lambda \rangle$, $\bigcap N_\lambda$ is also normal subgroup*
- *every subgroup of abelian group is normal*
- *factor group of abelian group is abelian*
- *factor group of cyclic group is cyclic*

Normalizers and centralizers

Definition 15. [normalizers and centralizers] *for subset $S \subset G$ of group G,*

$$\{x \in G | xSx^{-1} = S\}$$

is subgroup, called normalizer of S, *and also called* centralizer of a *when $S = \{a\}$ is singletone;*

$$\{x \in G | (\forall y \in S)(xyx^{-1} = y)\}$$

called centralizer of S, *and centralizer of G itself, called* center of G

- $e.g.$, $A \mapsto \det A$ of multiplicative group of square matrices in $\mathbf{R}^{n \times n}$ into $\mathbf{R} \sim \{0\}$ is homeomorphism, kernel of which called *special linear group,* and (of course) is normal

Normalizers and congruence

Proposition 6. [normalizers of groups] *subgroup $H \subset G$ of group G is normal subgroup of its normalizer N_H*

- *subgroup $H \subset G$ of group G is normal subgroup of its normalizer N_H*
- *subgroup $K \subset G$ with $H \subset K$ where H is normal in K is contained in N_H*
- *for subgroup $K \subset N_H$, KH is group and H is normal in KH*
- *normalizer of H is largest subgroup of G in which H is normal*

Definition 16. [congruence with respect to normal subgroup] *for normal subgroup $H \subset G$ of group G, we write*

$$x \equiv y \pmod{H}$$

if $xH = yH$, read x and y are congruent modulo H - this notation used mostly for additive groups

Exact sequences of homeomorphisms

Definition 17. [exact sequences of homeomorphisms] *below sequence of homeomorphisms with* $\mathrm{Im}\, f = \mathrm{Ker}\, g$

$$G' \xrightarrow{f} G \xrightarrow{g} G''$$

said to be exact

below sequence of homeomorphisms with $\mathrm{Im}\, f_i = \mathrm{Ker}\, f_{i+1}$

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \longrightarrow \cdots \xrightarrow{f_{n-1}} G_n$$

said to be exact

- for normal subgroup $H \subset G$ of group G, sequence $H \xrightarrow{j} G \xrightarrow{\varphi} G/H$ is exact where j is inclusion and φ
- $0 \to G' \xrightarrow{f} G \xrightarrow{g} G'' \to 0$ is exact *if and only if* f injective, g surjective, and $\mathrm{Im}\, f = \mathrm{Ker}\, g$

- if $H = \operatorname{Ker} g$ above, $0 \to H \to G \to G/H \to 0$
- more precisely, exists commutative diagram as in the figure, in which vertical mappings are isomorphisms and rows are *exact*

$$
\begin{tikzcd}
0 \arrow[r] & G' \arrow[r, "f"] \arrow[d] & G \arrow[r, "g"] \arrow[d] & G'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r] & G \arrow[r] & G/H \arrow[r] & 0
\end{tikzcd}
$$

Canonical homeomorphism examples

all homeomorphisms described below called *canonical*

- for two groups G & G' and homeomorphism $f : G \to G'$ whose kernel is H, exists unique homeomorphism $f_* : G/H \to G'$ with

$$f = f_* \circ \varphi$$

  where $\varphi : G \to G/H$ is canonical map, and f_* is injective
  - f_* can be defined by $xH \mapsto f(x)$
  - *f_* said to be induced by f*
  - f_* induces isomorphism $\lambda : G/H \to \operatorname{Im} f$
  - below sequence summarizes above statements

$$G \xrightarrow{\varphi} G/H \xrightarrow{\lambda} \operatorname{Im} f \xrightarrow{j} G$$

    where j is inclusion

- for group G, subgroup $H \subset G$, and homeomorphism $f : G \to G'$ whose kernel contains H, intersection of all normal subgroups containing H, N, which is the smallest normal subgroup containing H, is contained in $\mathrm{Ker}\, f$, *i.e.*, $N \subset \mathrm{Ker}\, f$, and exists unique homeomorphism, $f_* : G/N \to G'$ such that

$$f = f_* \circ \varphi$$

  where $\varphi : G \to G/H$ is canonical map
  - f_* can be defined by $xN \mapsto f(x)$
  - *f_* said to be induced by f*

- for subgroups of G, H and K with $K \subset H$, $xK \mapsto xH$ induces homeomorphism of G/K into G/H, whose kernel is $\{xK | x \in H\}$, thus *canonical isomorphism*

$$(G/K)/(H/K) \approx (G/K)$$

  this can be shown in the figure where rows are exact

$$
\begin{tikzcd}
0 \arrow[r] & H \arrow[r] \arrow[d, "\text{can}"] & G \arrow[r] \arrow[d, "\text{can}"] & G/H \arrow[r] \arrow[d, "\text{id}"] & 0 \\
0 \arrow[r] & H/K \arrow[r] & G/K \arrow[r] & G/H \arrow[r] & 0
\end{tikzcd}
$$

- for subgroup $H \subset G$ and $K \subset G$ with H contained in normalizer of K, $H \cap K$ is normal subgroup of H, $HK = KH$ is subgroup of G, exists surjective homeomorphism

$$H \to HK/K$$

with $x \mapsto xK$, whose kernel is $H \cap K$, hence *canonical isomorphism*

$$H/(H \cap K) \approx HK/K$$

- for group homeomorphism $f : G \to G'$, normal subgroup of G', H',

$$H = f^{-1}(H') \subset G$$

as shown in the figure,

$$
\begin{tikzcd}
G \arrow[r] & G' \\
f^{-1}(H') \arrow[u] \arrow[r] & H' \arrow[u]
\end{tikzcd}
$$

H is normal in G and kernel of homeomorphism

$$G \xrightarrow{f} G' \xrightarrow{\varphi} G'/H'$$

is H where φ is canonical map, hence we have injective homeomorphism

$$\bar{f} : G/H \to G'/H'$$

again called *canonical homeomorphism*, giving commutative diagram in the figure; if f is surjective, $\bar{f}$ is isomorphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow f & & \downarrow \bar{f} & & \\
0 & \longrightarrow & H' & \longrightarrow & G' & \longrightarrow & G'/H' & \longrightarrow & 0
\end{array}$$

Towers

Definition 18. [towers of groups] *for group G, sequence of subgroups*

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_m$$

called tower of subgroups

- *said to be* normal *if every G_{i+1} is normal in G_i*
- *said to be* abelian *if normal and every factor group G_i/G_{i+1} is abelian*
- *said to be* cyclic *if normal and every factor group G_i/G_{i+1} is cyclic*

Proposition 7. [towers inded by homeomorphism] *for group homeomorphism $f : G \to G'$ and normal tower*

$$G' = G'_0 \supset G'_1 \supset G'_2 \supset \cdots \supset G'_m$$

tower

$$f^{-1}(G') = f^{-1}(G'_0) \supset f^{-1}(G'_1) \supset f^{-1}(G'_2) \supset \cdots \supset f^{-1}(G'_m)$$

is

- *normal if G'_i form normal tower*
- *abelian if G'_i form abelian tower*
- *cyclic if G'_i form cyclic tower*

because every homeomorphism

$$G_i/G_{i+1} \to G'_i/G'_{i+1}$$

is injective

Refinement of towers and solvability of groups

Definition 19. [refinement of towers] *for tower of subgroups, tower obtained by inserting finite number of subgroups, called* refinement of tower

Definition 20. [solvable groups] *group having an abelian tower whose last element is trivial subgroup, said to be* solvable

Proposition 8. [finite solvable groups]

- *abelian tower of finite group admits cyclic refinement*
- *finite solvable group admits cyclic tower, whose last element is trivial subgroup*

Theorem 1. [Feit-Thompson theorem] *group whose order is prime power is solvable*

Theorem 2. [solvability condition in terms of normal subgroups] *for group G and its normal subgroup H, G is solvable* if and only if *both H and G/H are solvable*

Commutators and commutator subgroups

Definition 21. [commutator] *for group G, $xyx^{-1}y^{-1}$ for $x, y \in G$, called* commutator

Definition 22. [commutator subgroups] *subgroup generated by commutators of group G, called* commutator subgroup, *denoted by* G^C, $i.e.$

$$G^C = \langle \{xyx^{-1}y^{-1} | x, y \in G\} \rangle$$

- G^C is normal in G
- G/G^C is commutative
- G^C is contained in kernel of every homeomorphism of G into commutative group
- (proof can be found in Proof 2) of above statements

- *commutator group is at the heart of solvability and non-solvability problems!*

Simple groups

Definition 23. [simple groups] *non-trivial group having no normal subgroup other than itself and trivial subgroup, said to be* simple

Proposition 9. [simple groups] *abelian group is simple* if and only if *cycle of prime order*

Butterfly lemma

Lemma 1. [butterfly lemma - Zassenhaus] *for subgroups U and V of a group and normal subgroups u and v of U and V respectively,*

$$u(U \cap v) \text{ is normal in } u(U \cap V)$$

$$(u \cap V)v \text{ is normal in } (U \cap V)v$$

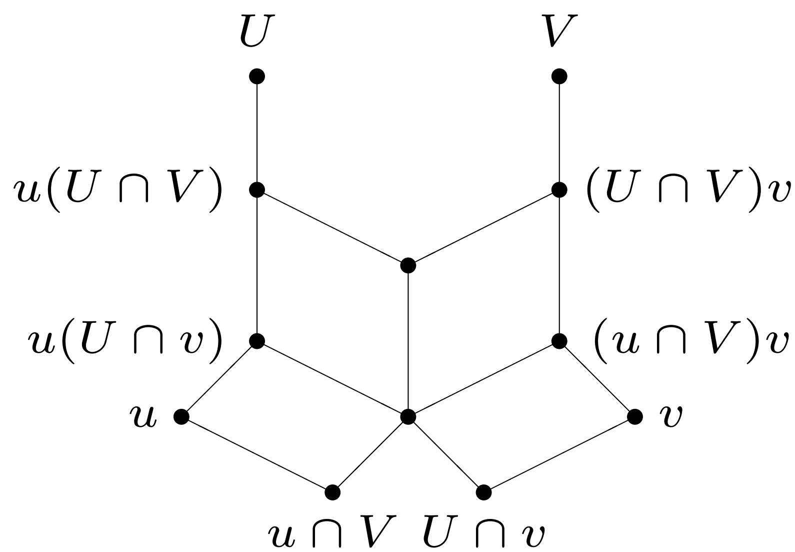
and factor groups are isomorphic, $i.e.$,

$$u(U \cap V)/u(U \cap v) \approx (U \cap V)v/(u \cap V)v$$

these shown in the figure

- indeed

$$(U \cap V)/((u \cap V)(U \cap v)) \approx u(U \cap V)/u(U \cap v) \approx (U \cap V)v/(u \cap V)v$$


U
V
$u(U \cap V)$
$(U \cap V)v$
$u(U \cap v)$
$(u \cap V)v$
u
v
$u \cap V$
$U \cap v$

Equivalent towers

Definition 24. [equivalent towers] *for two normal towers of same height starting from same group ending with trivial subgroup*

$$G = G_1 \supset G_2 \supset G_3 \supset \cdots \supset G_{n+1} = \{e\}$$

$$G = H_1 \supset H_2 \supset H_3 \supset \cdots \supset H_{n+1} = \{e\}$$

with

$$G_i/G_{i+1} \approx H_{\pi(i)+1}/H_{\pi(i)}$$

for some permutation $\pi \in \mathrm{Perm}(\{1,\ldots,n\})$, $i.e.$, *sequences of factor groups are same up to isomorphisms and permutation of indices, said to be* equivalent

Schreier and Jordan-Hölder theorems

Theorem 3. [Schreier theorem] *two normal towers starting from same group and ending with trivial subgroup have equivalent refinement*

Theorem 4. [Jordan-Hölder theorem] *all normal towers starting from same group and ending with trivial subgroup where each factor group is non-trivial and simple are equivalent*

Cyclic groups

Definition 25. [exponent of groups and group elements] *for group G, $n \in \mathbf{N}$ with $a^n = e$ for $a \in G$, called* exponent of a*; $n \in \mathbf{N}$ with $x^n = e$ for every $x \in G$, called* exponent of G

Definition 26. [period of group elements] *for group G and $a \in G$, smallest $n \in \mathbf{N}$ with $a^n = e$, called* period of a

Proposition 10. [period of elements of finite groups] *for finite group G of order $n > 1$, period of every non-unit element a ($\neq e$) devided n; if n is prime number, G is cyclic and period of every generator is n*

Proposition 11. [subgroups of cyclic groups] *every subgroup of cyclic group is cyclic and image of every homeomorphism of cyclic group is cyclic*

Properties of cyclic groups

Proposition 12. [properties of cyclic groups]

- *infinity cyclic group has exactly two generators; if a is one, a^{-1} is the other*
- *for cyclic group G of order n and generator x, set of generators of G is*

$$\{x^m | m \text{ is relatively prime to } n\}$$

- *for cyclic group G and two generators a and b, exists automorphism of G mapping a onto b; conversely, every automorphism maps a to some generator*
- *for cyclic group G of order n and $d \in \mathbf{N}$ dividing n, exists unique subgroup of order d*
- *for cyclic groups G_1 and G_2 of orders n and m respectively with n and m relatively prime, $G_1 \times G_2$ is cyclic group*
- *for non-cyclic finite abelian group G, exists subgroup isomorphic to $C \times C$ with C cyclic with prime order*

Symmetric groups and permutations

Definition 27. [symmetric groups and permutations] *for nonempty set S, group G of bijective functions of S onto itself with law of composition being function composition, called* symmetric group of S, *denoted by* $\mathrm{Perm}(S)$; *elements in* $\mathrm{Perm}(S)$ *called* permutations of S; *element swapping two disjoint elements in S leaving every others left, called* transposition

Proposition 13. [sign homeomorphism of finite symmetric groups] *for finite symmetric group S_n, exits unique homeomorphism $\epsilon : S_n \to \{-1, 1\}$ mapping every transposition, τ, to -1, $i.e.$, $\epsilon(\tau) = -1$*

Definition 28. [alternating groups] *element of finite symmetric group σ with $\epsilon(\sigma) = 1$, called* even, *element σ with $\epsilon(\sigma) = -1$, called* odd; *kernel of ϵ, called* alternating group, *denoted by* A_n

Theorem 5. [solvability of finite symmetric groups] *symmetric group S_n with $n \geq 5$ is* not *solvable*

Theorem 6. [simplicity of alternating groups] *alternating group A_n with $n \geq 5$ is simple*

Operations of group on set

Definition 29. [operations of group on set] *for group G and set S, homeomorphism*

$$\pi : G \to \mathrm{Perm}(S)$$

called operation of G on S *or* action of G on S

- *S, called* G-set
- *denote $\pi(x)$ for $x \in G$ by π_x, hence homeomorphism denoted by $x \mapsto \pi_x$*

- obtain mapping from such operation, $G \times S \to S$, with $(x, s) \mapsto \pi_x(s)$
- often abbreviate $\pi_x(s)$ by xs, with which the following two properties satisfied
  - $(\forall x, y \in G, s \in S)\ (x(ys) = (xy)s)$
  - $(\forall s \in S)\ (es = s)$
- conversely, for mapping $G \times S \to S$ with $(x, s) \mapsto xs$ satisfying above two properties, $s \mapsto xs$ is permutation for $x \in G$, hence π_x is homeomorphism of G into $\mathrm{Perm}(S)$
- thus, operation of G on S can be defined as mapping $S \times G \to S$ satisfying above two properties

Conjugation

Definition 30. [conjugation of groups] *for group G and map $\gamma_x : G \to G$ with $\gamma_x(y) = xyx^{-1}$, homeomorphism*

$$G \to \mathrm{Aut}(G) \text{ defined by } x \mapsto \gamma_x$$

called conjugation, which is operation of G on itself

- γ_x, called *inner*
- kernel of conjugation is *center of G*
- to avoid confusion, instead of writing xy for $\gamma_x(y)$, write

$$\gamma_x(y) = xyx^{-1} = {}^x y \text{ and } \gamma_{x^{-1}}(y) = x^{-1}yx = y^x$$

- for subset $A \subset G$, map $(x, A) \mapsto xAx^{-1}$ is operation of G on set of subsets of G
- similarly for subgroups of G
- two subsets of G, A and B with $B = xAx^{-1}$ for some $x \in G$, said to be *conjugate*

Translation

Definition 31. [translation] *operation of G on itself defined by map*

$$(x, y) \mapsto xy$$

called translation, *denoted by $T_x : G \to G$ with $T_x(y) = xy$*

- for subgroup $H \subset G$, $T_x(H) = xH$ is left coset
  - denote set of left cosets also by G/H even if H is not normal
  - denote set of right cosets also by $H\backslash G$
- examples of translation
  - $G = GL(V)$, group of linear automorphism of vector space with field F, for which, map $(A, v) \mapsto Av$ for $A \in G$ and $v \in V$ defines operation of G on V
    - G is subgroup of group of permutations, $\mathrm{Perm}(V)$
  - for $V = F^n$, G is group of nonsingular n-by-n matrices

Isotropy

Definition 32. [isotropy] *for operation of group G on set S*

$$\{x \in G | xs = s\}$$

called isotropy of G, *denoted by* G_s, *which is subgroup of* G

- for conjugation operation of group G, G_s is normalizer of $s \in G$
- isotropy groups are conjugate, $e.g.$, for $s, s' \in S$ and $y \in G$ with $ys = s'$,

$$G_{s'} = yG_sy^{-1}$$

- by definition, kernel of operation of G on S is

$$K = \bigcap_{s \in S} G_s \subset G$$

- operation with trivial kernel, said to be *faithful*
- $s \in G$ with $G_s = G$, called *fixed point*

Orbits of operation

Definition 33. [orbits of operation] *for operation of group G on set S, $\{xs|x \in G\}$, called* orbit of s under G*, denoted by* Gs

- for $x, y \in G$ in same coset of G_s, $xs = ys$, $i.e.$ $(\exists z \in G)\,(x, y \in zG_s) \Leftrightarrow xs = ys$
- hence, mapping $G/G_s \to S$ with $x \mapsto xG_s$ is morphism of G-sets, thus

Proposition 14. *for group G, operating on set S and $s \in S$, order of orbit Gs is equal to index $(G : G_s)$*

Proposition 15. *for subgroup H of group G, number of conjugate subgroups to H is index of normalizer of H in G*

Definition 34. [transitive operation] *operation with one orbit, said to be* transitive

Orbit decomposition and class formula

- orbits are disjoint

$$S = \coprod_{\lambda \in \Lambda} G s_\lambda$$

where s_λ are elements of distinct orbits

Formula 1. [orbit decomposition formula] *for group G operating on set S, index set Λ whose elements represent distinct orbits*

$$|S| = \sum_{\lambda \in \Lambda} (G : G_\lambda)$$

Formula 2. [class formula] *for group G and set $C \subset G$ whose elements represent distinct conjugacy classes*

$$(G : 1) = \sum_{x \in C} (G : G_x)$$

Sylow subgroups

Definition 35. [sylow subgroups] *for prime number p, finite group with order p^n for some $n \geq 0$, called p-group; subgroup $H \subset G$ of finite group G with order p^n for some $n \geq 0$, called p-subgroup; subgroup of order p^n where p^n is highest power of p dividing order of G, called p-Sylow subgroup*

Lemma 2. *finite abelian group of order divided by prime number p has subgroup of order p*

Theorem 7. [p-Sylow subgroups of finite groups] *finite group of order divided by prime number p has p-Sylow subgroup*

Lemma 3. [number of fixed points of group operations] *for p-group H, operating on finite set S*

- *number of fixed points of H is congruent to size of S modulo p, $i.e.$*

$$\#\text{ fixed points of } H \equiv |S| \pmod{p}$$

- *if H has exaxctly one fixed point, $|S| \equiv 1 \pmod{p}$*
- *if p divides $|S|$, $|S| \equiv 0 \pmod{p}$*

Sylow subgroups and solvability

Theorem 8. [solvability of finite p-groups] *finite p-group is solvable; if it is non-trivial, it has non-trivial center*

Corollary 1. *for non-trivial p-group, exists sequence of subgroups*

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n = G$$

where G_i is normal in G and G_{i+1}/G_i is cyclic group of order p

Lemma 4. [normality of subgroups of order p] *for finite group G and smallest prime number dividing order of G p, every subgroup of index p is normal*

Proposition 16. [solvability of groups of order pq] *group of order pq with p and q being distinct prime numbers, is solvable*

- now can prove following
  - group of order, 35, is solvable - implied by Proposition 8 and Proposition 12
  - group of order less than 60 is solvable

Rings

Rings

Definition 36. [ring] *set A together with two laws of composition called multiplication and addition which are written as product and sum respectively, satisfying following conditions, called* ring

- *A is commutative group with respect to addition - unit element denoted by 0*
- *A is monoid with respect to multiplication - unit element denoted by 1*
- *multiplication is distributive over addition, $i.e.$*

$$(\forall x, y, z \in A)\,((x+y)z = xz + yz \;\&\; z(x+y) = zx + zy)$$

do not assume $1 \neq 0$

- can prove, $e.g.$,
  - $(\forall x \in A)\,(0x = 0)$ because $0x + x = 0x + 1x = (0+1)x = 1x = x$
  - if $1 = 0$, $A = \{0\}$ because $x = 1x = 0x = 0$
  - $(\forall x, y \in A)\,((-x)y = -(xy))$ because $xy + (-x)y = (x + -x)y = 0y = 0$

Definition 37. [subring] *subset of ring which itself is ring with same additive and multiplicative laws of composition, called* subring

More on ring

Definition 38. [multiplicative group of invertible elements of ring] *subset U of ring A such that every element of U has both left and right inverses, called* group of units of A *or* group of invertible elements of A, *sometimes denoted by* A^*

Definition 39. [division ring] *ring with $1 \neq 0$ and every nonzero element being invertible, called* division ring

Definition 40. [commutative ring] *ring A with $(\forall x, y \in A)\,(xy = yx)$, called* commutative ring

Definition 41. [center of ring] *subset $C \subset A$ of ring A such that*

$$C = \{a \in A | \forall x \in A, xa = ax\}$$

is subring, *and is called* center of ring A

Fields

Definition 42. [field] *commutative division ring, called* field

General distributivity

- general distributivity - for ring A, $\langle x_i \rangle_{i=1}^n \subset A$ and $\langle y_i \rangle_{i=1}^n \subset A$

$$\left(\sum x_i\right)\left(\sum y_j\right) = \sum_i \sum_j x_i y_j$$

Ring examples

- for set S and ring A, *set of all mappings of S into A* $\mathrm{Map}(S, A)$ whose addition and multiplication are defined as below, is *ring* (proof can be found in Proof 3)

$$(\forall f, g \in \mathrm{Map}(S, A))\,(\forall x \in S)\,((f + g)(x) = f(x) + g(x))$$

$$(\forall f, g \in \mathrm{Map}(S, A))\,(\forall x \in S)\,((fg)(x) = f(x)g(x))$$

  - additive and multiplicative unit elements of $\mathrm{Map}(S, A)$ are constant maps whose values are additive and multiplicative unit elements of A respectively
  - $\mathrm{Map}(S, A)$ is commutative *if and only if* A is commutative
  - for set S, $\mathrm{Map}(S, \mathbf{R})$ (page 2) is a commutative ring
- for abelian group M, *set $\mathrm{End}(M)$ of group homeomorphisms of M into itself* is *ring* with normal addition and mapping composition as multiplication (proof can be found in Proof 4)
  - additive and multiplicative unit elements of $\mathrm{End}(M)$ are constant map whose value is the unit element of M and identity mapping respectively

- not commutative in general

- for ring A, *set $A[X]$ of polynomials over A* is *ring*, (Definition 70)

- for field K, $K^{n\times n}$, $i.e.$, set of n-by-n matrices with components in K, is *ring*

  - $\left(K^{n\times n}\right)^*$, $i.e.$, multiplicative group of units of $K^{n\times n}$, consists of non-singular matrices, $i.e.$, those whose determinants are nonzero

Group ring

Definition 43. [group ring] *for group G and field K, set of all formal linear combinations $\sum_{x\in G} a_x x$ with $a_x \in K$ where a_x are zero except finite number of them where addition is defined normally and multiplication is defined as*

$$\left(\sum_{x\in G} a_x x\right)\left(\sum_{y\in G} b_y y\right) = \sum_{z\in G}\left(\sum_{xy=z} a_x b_y xy\right)$$

called group ring, denoted by $K[G]$

- $\sum_{xy=z} a_x b_y$ *above defines what is called convolution product*

Convolution product

Definition 44. [convolution product] *for two functions f, g on group G, convolution (product), denoted by $f * g$, defined by*

$$(f * g)(z) = \sum_{xy=z} f(x)f(y)$$

as function on group G

- *one may restrict this definition to functions which are 0 except at finite number of elements*

- for $f, g \in L^1(\mathbf{R})$, can define *convolution product* $f * g$ by

$$(f * g)(x) = \int_{\mathbf{R}} f(x-y)g(y)dy$$

  – satisfies all axioms of ring except that there is not unit element

– commutative (essentially because **R** is commutative)

- more generally, for locally compact group G wiht Haar measure μ, can define *convolution product* by

$$(f * g)(x) = \int_G f(xy^{-1})g(y)d\mu(y)$$

Ideals of ring

Definition 45. [ideal] *subset $\mathfrak{a}$ of ring A which is subgroup of additive group of A with $A\mathfrak{a} \subset \mathfrak{a}$, called* left ideal*; indeed, $A\mathfrak{a} = \mathfrak{a}$ because A has 1;* right ideal *can be similarly defined, $i.e.$, $\mathfrak{a}A = \mathfrak{a}$; subset which is both left and right ideal, called* two-sided ideal *or simply* ideal

- for ring A, (0) are A itself area ideals

Definition 46. [principal ideal] *for ring A and $a \in A$, left ideal Aa, called* principal left ideal

- *a, said to be generator of $\mathfrak{a} = Aa$ (over A)*

Definition 47. [principal two-sided ideal] *AaA, called* principal two-sided ideal *where*

$$AaA = \bigcup_{i=1}^{\infty} \left\{ \sum_{i=1}^{n} x_i a y_i \middle| x_i, y_i \in A \right\}$$

Lemma 5. [ideals of field] *ideals of field only ideals of field are the field itself and zero ideal*

Principle rings

Definition 48. [principal ring] *commutative ring of which every ideal is principal and* $1 \neq 0$, *called* principal ring

- **Z** (set of integers) is *principal* ring (proof can be found in Proof 5)
- $k[X]$ (ring of polynomials) for field k is *principal* ring
- ring of algebraic integers in number field K is *not* necessarily principal
  - let $\mathfrak{p}$ be prime ideal, let $R_\mathfrak{p}$ be ring of all elements a/b with $a, b \in R$ and $b \notin \mathfrak{p}$, then $R_\mathfrak{p}$ is principal, with one prime ideal $\mathfrak{m}_\mathfrak{p}$ consisting of all elements a/b as above but with $a \in \mathfrak{p}$
- let A be set of entire functions on complex plane, then A is commutative ring, and every finitely generated ideal is *principal*
  - given discrete set of complex numbers $\{z_i\}$ and nonnegative integers $\{m_i\}$, exists entire function f having zeros at z_i of multiplicity m_i and *no* other zeros
  - every principal ideal is of form Af for some such f
  - group of units A^* in A consists of functions having no zeros

Ideals as both additive and multiplicative monoids

- ideals form additive monoid
  - for left ideals $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ of ring A, $\mathfrak{a}+\mathfrak{b}$ is left ideal, $(\mathfrak{a}+\mathfrak{b})+\mathfrak{c} = \mathfrak{a}+(\mathfrak{b}+\mathfrak{c})$, hence form additive monoid with (0) as the unit elemenet
  - similarly for right ideals & two-sided ideals
- ideals form multiplicative monoid
  - for left ideals $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ of ring A, define $\mathfrak{a}\mathfrak{b}$ as

$$\mathfrak{a}\mathfrak{b} = \bigcup_{i=1}^{\infty} \left\{ \sum_{i=1}^{n} x_i y_i \middle| x_i \in \mathfrak{a}, y_i \in \mathfrak{b} \right\}$$

    then $\mathfrak{a}\mathfrak{b}$ is also left ideal, $(\mathfrak{a}\mathfrak{b})\mathfrak{c} = \mathfrak{a}(\mathfrak{b}\mathfrak{c})$, hence form multiplicative monoid with A itself as the unit elemenet; for this reason, this unit element A, $i.e.$, the ring itself, often written as (1)
  - similarly for right ideals & two-sided ideals
- ideal multiplication is also distributive over addition
- however, set of ideals does *not* form ring (because the additive monoid is *not* group)

Generators of ideal

Definition 49. [generators of ideal] *for ring A and $a_1, \ldots, a_n \subset A$, set of elements of A of form*

$$\sum_{i=1}^{n} x_i a_i$$

with $x_i \in A$, is left ideal, denoted by $(a_1, \ldots, a_n)$, called generators of the left ideal; similarly for right ideals

- above equal to smallest ideals containing a_i, $i.e.$, intersection of all ideals containing a_i

$$\bigcap_{a_1,\ldots,a_n \in \mathfrak{a}} \mathfrak{a}$$

(proof can be found in Proof 6) - just like set (σ-)algebras in set theory on page **??**

Entire rings

Definition 50. [zero divisor] *for ring A, $x, y \in A$ with $x \neq 0$, $y \neq 0$, and $xy = 0$, said to be* zero divisors

Definition 51. [entire ring] *commutative ring with no zero divisors for which $1 \neq 0$, said to be* entire*; entire ring, sometimes called* integral domain

Lemma 6. [every field is entire ring] *every field is entire ring*

Ring-homeomorphism

Definition 52. [ring-homeomorphism] *mapping of ring into ring $f : A \to B$ such that f is monoid-homeomorphism for both additive and multiplicative structure on A and B, i.e.,*

$$(\forall a, b \in A)\,(f(a+b) = f(a) + f(b)\ \&\ f(ab) = f(a)f(b))$$

and

$$f(1) = 1\ \&\ f(0) = 0$$

called ring-homeomorphism*;* kernel*, defined to be kernel of f viewed as additive homeomorphism*

- *kernel of ring-homeomorphism $f : A \to B$* is ideal of A (proof can be found in Proof 7)
- conversely, for ideal $\mathfrak{a}$, can construct factor ring $A/\mathfrak{a}$
- simply say "homeomorphism" if reference to ring is clear

Proposition 17. [injectivity of field homeomorphism] *ring-homeomorphism from field into field is injective (due to Lemma 5)*

Factor ring and canonical map

Definition 53. [factor ring and residue class] *for ring A and an ideal $\mathfrak{a} \subset A$, set of cosets $x + \mathfrak{a}$ for $x \in A$ combined with* addition *defined by viewing A and $\mathfrak{a}$ as additive groups,* multiplication *defined by $(x + \mathfrak{a})(y + \mathfrak{a}) = xy + \mathfrak{a}$, which satisfy all requirements for ring, called* factor ring *or* residue class ring*, denoted by $A/\mathfrak{a}$; cosets in $A/\mathfrak{a}$, called* residue classes modulo $\mathfrak{a}$*, and each coset $x + \mathfrak{a}$ called* residue class of x modulo $\mathfrak{a}$

- for ring A and ideal $\mathfrak{a}$
  - for subset $S \subset \mathfrak{a}$, write $S \equiv 0 \pmod{\mathfrak{a}}$
  - for $x, y \in A$, if $x - y \in \mathfrak{a}$, write $x \equiv y \pmod{\mathfrak{a}}$
  - if $\mathfrak{a} = (a)$ for $a \in A$, for $x, y \in A$, if $x - y \in \mathfrak{a}$, write $x \equiv y \pmod{a}$

Definition 54. [canonical map of ring] *ring-homeomorphism of ring A into factor ring $A/\mathfrak{a}$*

$$A \to A/\mathfrak{a}$$

called canonical map of A into $A/\mathfrak{a}$

Factor ring induced ring-homeomorphism

Proposition 18. [factor ring induced ring-homeomorphism] *for ring-homeomorphism $g : A \to A'$ whose kernel contains ideal $\mathfrak{a}$, exists unique ring-homeomorphism $g_* : A/\mathfrak{a} \to A'$ making diagram in the figure commutative, $i.e.$, $g^* \circ f = g$ where f is the ring canonical map $f : A \to A/\mathfrak{a}$*

$$
\begin{tikzcd}
A \arrow[rr, "g"] \arrow[dr, "f"'] & & A' \\
& A/\mathfrak{a} \arrow[ur, "g^*"'] &
\end{tikzcd}
$$

- the ring canonical map $f : A \to A/\mathfrak{a}$ is universal in category of homeomorphisms whose kernel contains $\mathfrak{a}$

Prime ideal and maximal ideal

Definition 55. [prime ideal] *for commutative ring A, ideal $\mathfrak{p} \neq A$ with $A/\mathfrak{p}$ entire, called* prime ideal *or just* prime*;*

- equivalently, ideal $\mathfrak{p} \neq A$ is *prime if and only if* $(\forall x, y \in A)\,(xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ or $y \in \mathfrak{p})$

Definition 56. [maximal ideal] *for commutative ring A, ideal $\mathfrak{m} \neq A$ such that*

$$(\forall \textit{ ideal } \mathfrak{a} \subset A)\,(\mathfrak{m} \subset \mathfrak{a} \Rightarrow \mathfrak{a} = A)$$

called maximal ideal

Lemma 7. [properties of prime and maximal ideals] *for commutative ring A*

- *every maximal ideal is prime*
- *every ideal is contained in some maximal ideal*
- *ideal $\{0\}$ is prime* if and only if *A is entire*
- *ideal $\mathfrak{m}$ is maximal* if and only if *$A/\mathfrak{m}$ is field*
- *inverse image of prime ideal of commutative ring homeomorphism is prime*

Embedding of ring

Definition 57. [ring-isomorphism] *bijective ring-homeomorphism (Definition 52) is isomorphism*

- indeed, for bijective ring-isomorphism $f : A \to B$, exists set-theoretic inverse $g : B \to A$ of f, which is ring-homeomorphism

Lemma 8. [image of ring-homeomorphism is subring] *image $f(A)$ of ring-homeomorphism $f : A \to B$ is subring of B (proof can be found in Proof 8)*

Definition 58. [embedding of ring] *ring-isomorphism between A and its image, established by injective ring-homeomorphism $f : A \to B$, called* embedding of ring

Definition 59. [induced injective ring-homeomorphism] *for ring-homeomorphism $f : A \to A'$ and ideal $\mathfrak{a}'$ of A', injective ring-homeomorphism*

$$A/f^{-1}(\mathfrak{a}') \to A'/\mathfrak{a}'$$

called induced injective ring-homeomorphism

Characteristic of ring

- for ring A, consider ring-homeomorphism

$$\lambda : \mathbf{Z} \to A$$

such that

$$\lambda(n) = ne$$

where e is multiplicative unit element of A

  - kernel of λ is ideal (n) for some $n \geq 0$, $i.e.$, ideal generated by some nonnegative integer n
  - hence, canonical injective ring-homeomorphism $\mathbf{Z}/n\mathbf{Z} \to A$, which is ring-isomorphism between $\mathbf{Z}/n\mathbf{Z}$ and subring of A
  - when $n\mathbf{Z}$ is prime ideal, exist two cases; either $n = 0$ or $n = p$ for prime number p

Definition 60. [characteristic of ring] *ring A with $\{0\}$ as prime ideal kernel above, said to have characteristic 0; if prime ideal kernel is $p\mathbf{Z}$ for prime number p, A, said to have characteristic p, in which case, A contains (isomorphic image of) $\mathbf{Z}/p\mathbf{Z}$ as subring, abbreviated by $\mathbf{F}_p$*

Prime fields and prime rings

- field K has characteristic 0 or p for prime number p
- K contains as subfield (isomorphic image of)
  - **Q** if characteristic is 0
  - $\mathbf{F}_p$ if characteristic is p

Definition 61. [prime field] *in above cases, both* **Q** *and* $\mathbf{F}_p$*, called* prime field (contained in K)*; since prime field is smallest subfield of* K *containing* 1 *having no automorphism other than identity, identify it with* **Q** *or* $\mathbf{F}_p$ *for each case*

Definition 62. [prime ring] *in above cases,* prime ring (contained in K) *means either integers* **Z** *if* K *has characteristic* 0 *or* $\mathbf{F}_p$ *if* K *has characteristic* p

$\mathbf{Z}/n\mathbf{Z}$

- $\mathbf{Z}$ is ring
- every ideal of $\mathbf{Z}$ is principal, $i.e.$, either $\{0\}$ or $n\mathbf{Z}$ for some $n \in \mathbf{N}$ (refer to page 62)
- ideal of $\mathbf{Z}$ is prime *if and only if* is $p\mathbf{Z}$ for some prime number $p \in \mathbf{N}$
  - $p\mathbf{Z}$ is maximal ideal

Definition 63. [ring of integers modulo n] $\mathbf{Z}/n\mathbf{Z}$, *called* ring of integers modulo n*; abbreviated as* mod n

- $\mathbf{Z}/p\mathbf{Z}$ for prime p is *field* and denoted by $\mathbf{F}_p$

Euler phi-function

Definition 64. [Euler phi-function] *for $n > 1$, order of divison ring of $\mathbf{Z}/n\mathbf{Z}$, called* Euler phi-function*, denoted by $\varphi(n)$; if prime factorization of n is*

$$n = p_1^{e_1} \cdots p_r^{e_r}$$

with distinct p_i and $e_i \geq 1$

$$\varphi(n) = p_1^{e_1 - 1}(p_1 - 1) \cdots p_r^{e_r - 1}(p_r - 1)$$

Theorem 9. [Euler's theorem] *for x prime to n*

$$x^{\varphi(n)} \equiv 1 \pmod{n}$$

Chinese remainder theorem

Theorem 10. [Chinese remainder theorem] *for ring A and n ideals $\mathfrak{a}_1$, . . . $\mathfrak{a}_n$ ($n \geq 2$) with $\mathfrak{a}_i + \mathfrak{a}_j = A$ for all $i \neq j$*

$$(\forall x_1, \ldots, x_n \in A)\,(\exists x \in A)\,(\forall 1 \leq i \leq n)\,(x \equiv x_i \pmod{\mathfrak{a}_i})$$

Corollary 2. [isomorphism induced by Chinese remainder theorem] *for ring A, n ideals $\mathfrak{a}_1$, . . . $\mathfrak{a}_n$ ($n \geq 2$) with $\mathfrak{a}_i + \mathfrak{a}_j = A$ for all $i \neq j$, and map of A into product induced by canonical maps of A onto $A/\mathfrak{a}_i$ for each factor, $i.e.$,*

$$f : A \to \prod A/\mathfrak{a}_i$$

f is surjective and $\mathrm{Ker}\, f = \bigcap \mathfrak{a}_i$, hence, exists isomorphism

$$A/\cap\,\mathfrak{a}_i \approx \prod A/\mathfrak{a}_i$$

Isomorphism of endomorphisms of cyclic groups

Theorem 11. [isomorphism of endomorphisms of cyclic groups] *for cyclic group A of order n, endomorphisms of A into A with $x \mapsto kx$ for $k \in \mathbf{Z}$ induce*

- *ring isomorphism*

$$\mathbf{Z}/n\mathbf{Z} \approx \mathrm{End}(A)$$

- *group isomorphism*

$$(\mathbf{Z}/n\mathbf{Z})^* \approx \mathrm{Aut}(A)$$

where $(\mathbf{Z}/n\mathbf{Z})^*$ denotes group of units of $\mathbf{Z}/n\mathbf{Z}$ (Definition 38)

- $e.g.$, for group of n-th roots of unity in $\mathbf{C}$, all automorphisms are given by

$$\xi \mapsto \xi^k$$

for $k \in (\mathbf{Z}/n\mathbf{Z})^*$

Irreducibility and factorial rings

Definition 65. [irreducible ring element] *for entire ring A, non-unit non-zero element $a \in A$ with*

$$(\forall b, c \in A) \, (a = bc \Rightarrow b \text{ or } c \text{ is unit})$$

said to be irreducible

Definition 66. [unique factorization into irreducible elements] *for entire ring A, element $a \in A$ for which, exists unit u and irreducible elements, p_1, . . . , p_r in A such that*

$$a = u \prod p_i$$

and this expression is unique up to permutation and multiplications by units, said to have unique factorization into irreducible elements

Definition 67. [factorial ring] *entire ring with every non-zero element has unique factorial into irreducible elements, called* factorial ring *or* unique factorization ring

Greatest common divisor

Definition 68. [devision of entire ring elements] *for entire ring A and nonzero elements $a, b \in A$,* a said to divide b *if exists $c \in A$ such that $ac = b$, denoted by* $a|b$

Definition 69. [greatest common divisor] *for entire ring A and $a, b \in A$, $d \in A$ which divides a and b and satisfies*

$$(\forall c \in A)\,(c|a \;\&\; c|b \Rightarrow c|d)$$

called greatest common divisor (g.c.d.) of a and b

Proposition 19. [existence of greatest common divisor of principal entire rings] *for principal entire ring A and nonzero $a, b \in A$, $c \in A$ with $(a, b) = (c)$ is g.c.d. of a and b*

Theorem 12. [principal entire ring is factorial] *principal entire ring is factorial*

Polynomials

Why (ring of) polynomials?

- lays ground work for polynomials in general
- needs polynomials over arbitrary rings for diverse purposes
  - polynomials over finite field which cannot be identified with polynomial functions in that field
  - polynomials with integer coefficients; reduce them mod p for prime p
  - polynomials over arbitrary commutative rings
  - rings of polynomial differential operators for algebraic geometry & analysis
- $e.g.$, ring learning with errors (RLWE) for cryptographic algorithms

Ring of polynomials

- exist many ways to define polynomials over commutative ring; here's one

Definition 70. [polynomial] *for ring A, set of functions from monoid $S = \{X^r | r \in \mathbf{Z}, r \geq 0\}$ into A which are equal to 0 except finite number of elements of S, called* polynomials over A, *denoted by* $A[X]$

- for every $a \in A$, define function which has value a on X^n, and value 0 for every other element of S, by aX^r
- then, *a* polynomial can be uniquely written as

$$f(X) = a_0 X^0 + \cdots + a_n X^n$$

  for some $n \in \mathbf{Z}_+$, $a_i \in A$

- a_i, called *coefficients of f*

Polynomial functions

Definition 71. [polynomial function] *for two rings A and B with $A \subset B$ and $f \in A[X]$ with $f(X) = a_0 + a_1 X + \cdots + a_n X^n$, map $f_B : B \to B$ defined by*

$$f_B(x) = a_0 + a_1 x + \cdots + a_n x^n$$

called polynomial function associated with $f(X)$

Definition 72. [evaluation homeomorphism] *for two rings A and B with $A \subset B$ and $b \in B$, ring homeomorphism from $A[X]$ into B with association, $\mathrm{ev}_b : f \mapsto f(b)$, called* evaluation homeomorphism*, said to be obtained by* substituting b for X in f

- hence, for $x \in B$, subring $A[x]$ of B generated by x over A is ring of all polynomial values $f(x)$ for $f \in A[X]$

Definition 73. [variables and transcendentality] *for two rings A and B with $A \subset B$, if $x \in B$ makes evaluation homeomorphism $\mathrm{ev}_x : f \mapsto f(x)$ isomorphic, x, said to be* transcendental over A *or* variable over A

- in particular, X *is variable over A*

Polynomial examples

- consider $\alpha = \sqrt{2}$ and $\{a + b\alpha \,|a, b \in \mathbf{Z}\}$, subring of $\mathbf{Z}[\alpha] \subset \mathbf{R}$ generated by α.
  - α is *not* transcendental because $f(\alpha) = 0$ for $f(X) = X^2 - 1$
  - hence kernel of evaluation map of $\mathbf{Z}[X]$ into $\mathbf{Z}[\alpha]$ is not injective, hence not isomorphism
  - indeed
$$\mathbf{Z}[\alpha] = \{a + b\alpha \,|a, b \in \mathbf{Z}\}$$

- consider $\mathbf{F}_p$ for prime number p
  - $f(X) = X^p - X \in \mathbf{F}_p[X]$ is not zero polynomial, but because $x^{p-1} \equiv 1$ for every nonzero $x \in \mathbf{F}_p$ by Theorem 9 (Euler's theorem), $x^p \equiv x$ for every $x \in \mathbf{F}_p$, thus for polynomial function, $f_{\mathbf{F}_p}$, $f_{\mathbf{F}_p}(x) = 0$ for every x in $\mathbf{F}_p$
  - $i.e.$, *non-zero polynomial induces zero polynomial function*

Reduction map

- for homeomorphism $\varphi : A \to B$ of commutative rings, exists associated homeomorphisms of polynomial rings $A[X] \to B[X]$ such that

$$f(X) = \sum a_i X^i \mapsto \sum \varphi(a_i) X^i = (\varphi f)(X)$$

Definition 74. [reduction map] *above ring homeomorphism $f \mapsto \varphi f$, called* reduction map

- $e.g.$, for complex conjugate $\varphi : \mathbf{C} \to \mathbf{C}$, homeomorphism of $\mathbf{C}[X]$ into itself can be obtained by reduction map $f \mapsto \varphi f$, which is complex conjugate of polynomials with complex coefficients

Definition 75. [reduction of f modulo p] *for prime ideal $\mathfrak{p}$ of ring A and surjective canonical map $\varphi : A \to A/\mathfrak{p}$, reduction map φf for $f \in A[X]$, sometimes called* reduction of f modulo $\mathfrak{p}$

Basic properties of polynomials in one variable

Theorem 13. [Euclidean algorithm] *for set of all polynomials in one variable of nonnegative degrees $A[X]$ with commutative ring A*

$$(\forall f, g \in A[X] \text{ with leading coefficients of } g \text{ unit in } A)$$
$$(\exists q, r \in A[X] \text{ with } \deg r < \deg g)\,(f = qg + r)$$

Theorem 14. [principality of polynomial ring] *polynomial ring in one variable $k[X]$ with field k is principal*

Corollary 3. [factoriality of polynomial ring] *polynomial ring in one variable $k[X]$ with field k is factorial*

Constant, monic, and irreducible polynomials

Definition 76. [constant and monic polynomials] *$k \in k[X]$ with field k, called* constant polynomial*; $f(x) \in k[X]$ with leading coefficient 1, called* monic polynomial

Definition 77. [irreducible polynomials] *polynomial $f(x) \in k[X]$ such that*

$$(\forall g(X), h(X) \in k[X])\, (f(X) = g(X)h(X) \Rightarrow g(X) \in k \text{ or } h(X) \in k)$$

said to be irreducible

Roots or zeros of polynomials

Definition 78. [root of polynomial] *for commutative ring B, its subring $A \subset B$, and $f(x) \in A[X]$ in one variable, $b \in B$ satisfying*

$$f(b) = 0$$

called root *or* zero *of f*

Theorem 15. [number of roots of polynomial] *for field k, polynomial $f \in k[X]$ in one variable of degree $n \geq 0$ has at most n roots in k; if a is root of f in k, $X - a$ divides $f(X)$*

Induction of zero functions

Corollary 4. [induction of zero function in one variable] *for field k and infinite subset $T \subset k$, if polynomial $f \in k[X]$ in one variable over k satisfies*

$$(\forall a \in k) \, (f(a) = 0)$$

then $f(0) = 0$, $i.e.$, f induces zero function

Corollary 5. [induction of zero function in multiple variables] *for field k and n infinite subsets of k, $\langle S_i \rangle_{i=1}^{n}$, if polynomial in n variables over field k satisfies*

$$(\forall a_i \in S_i \text{ for } 1 \leq i \leq n) \, (f(a_1, \ldots, a_n) = 0)$$

then $f = 0$, $i.e.$, f induces zero function

Corollary 6. [induction of zero functions in multiple variables - infinite fields] *if polynomial in n variables over infinite field k induces zero function in $k^{(n)}$, $f = 0$*

Corollary 7. [induction of zero functions in multiple variables - finite fields] *if polynomial in n variables over finite field k of order q, degree of which in each variable is less than q, induces zero function in $k^{(n)}$, $f = 0$*

Reduced polynomials and uniqueness

- for field k with q elements, polynomial in n variables over k can be expressed as

$$f(X_1, \ldots, X_n) = \sum a_i X_1^{\nu_{i,1}} \cdots X_n^{\nu_{i,n}}$$

  for finite sequence, $\langle a_i \rangle_{i=1}^m$, and $\langle \nu_{i,1} \rangle_{i=1}^m$, . . . , $\langle \nu_{i,n} \rangle_{i=1}^m$ where $a_i \in k$ and $\nu_{i,j} \geq 0$
- because $X_i^q = X_i$ for any X_i, any $\nu_{i,j} \geq q$ can be (repeatedly) replaced by $\nu_{i,j} - (q-1)$, hence f can be rewritten as

$$f(X_1, \ldots, X_n) = \sum a_i X_1^{\mu_{i,1}} \cdots X_n^{\mu_{i,n}}$$

  where $0 \leq \mu_{i,j} < q$ for all i, j

Definition 79. [reduced polynomials] *above polynomial, called reduced polynomial, denoted by f^**

Corollary 8. [uniqueness of reduced polynomials] *for field k with q elements, reduced polynomial is unique (by Corollary 7)*

Multiplicative subgroups and n-th roots of unity

Definition 80. [multiplicative subgroup of field] *for field k, subgroup of group $k^* = k \sim \{0\}$, called* multiplicative subgroup of k

Theorem 16. [finite multiplicative subgroup of field is cyclic] *finite multiplicative subgroup of field is cyclic*

Corollary 9. [multiplicative subgroup of finite field is cyclic] *multiplicative subgroup of finite field is cyclic*

Definition 81. [primitive n-th root of unity] *generator for group of n-th roots of unity, called* primitive n-th root of unity*; group of roots of unity, denoted by μ; group of roots of unity in field k, denoted by $\mu(k)$*

Algebraic closedness

Definition 82. [algebraically closed] *field k, for which every polynomial in $k[X]$ of positive degree has root in k, said to be* algebraically closed

- $e.g.$, complex numbers are algebraically closed
- every field is contained in some algebraically closed field (Theorem 17)
- for algebraically closed field k
  - (of course) every irreducible polynomial in $k[X]$ is of degree 1
  - unique factorization of polynomial of nonnegative degree can be written in form

$$f(X) = c \prod_{i=1}^{r} (X - \alpha_i)^{m_i}$$

with nonzero $c \in k$, distinct roots, $\alpha_1, \ldots, \alpha_r \in k$, and $m_1, \ldots, m_r \in \mathbf{N}$

Derivatives of polynomials

Definition 83. [derivative of polynomial over commutative ring] *for polynomial* $f(X) = a_n X^n + \cdots + a_1 X + a_0 \in A[X]$ *with commutative ring* A, *map* $D : A[X] \to A[X]$ *defined by*

$$Df(X) = n a_n X^{n-1} + \cdots + a_1$$

called derivative of polynomial, *denoted by* $f'(X)$;

- for $f, g \in A[X]$ with commutative ring A, and $a \in A$

$$(f+g)' = f' + g' \quad \& \quad (fg)' = f'g + fg' \quad \& \quad (af)' = af'$$

Multiple roots and multiplicity

- nonzero polynomial $f(X) \in k[X]$ in one variable over field k having $a \in k$ as root can be written of form
$$f(X) = (X-a)^m g(X)$$
with some polynomial $g(X) \in A[X]$ relatively prime to $(X-a)$ (hence, $g(a) \neq 0$)

Definition 84. [multiplicity and multiple roots] *above, m, called* multiplicity of a in f; *a, said to be* multiple root of f *if $m > 1$*

Proposition 20. [necessary and sufficient condition for multiple roots] *for polynomial f of one variable over field k, $a \in k$ is multiple root of f* if and only if *$f(a) = 0$ and $f'(a) = 0$*

Proposition 21. [derivative of polynomial] *for polynomial $f \in K[X]$ over field K of positive degree, $f' \neq 0$ if K has characteristic 0; if K has characteristic $p > 0$, $f' = 0$* if and only if
$$f(X) = \sum_{\nu=1}^{n} a_\nu X^\nu$$
where p divides each integer ν whenever $a_\nu \neq 0$

Frobenius endomorphism

- homeomorphism of K into itself $x \mapsto x^p$ has trivial kernel, hence injective

- hence, iterating $r \geq 1$ times yields endomorphism, $x \mapsto x^{p^r}$

Definition 85. [Frobenius endomorphism] *for field K, prime number p, and $r \geq 1$, endomorphism of K into itself, $x \mapsto x^{p^r}$, called* Frobenius endomorphism

Roots with multiplicity p^r in fields having characteristic p

- for field K having characteristic p
  - $p|\binom{p}{\nu}$ for all $0 < \nu < p$ because p is prime, hence, for every $a, b \in K$

$$(a+b)^p = a^p + b^p$$

  - applying this resurvely r times yields

$$(a+b)^{p^r} = (a^p + b^p)^{p^{r-1}} = (a^{p^2} + b^{p^2})^{p^{r-2}} = \cdots = a^{p^r} + b^{p^r}$$

    hence

$$(X-a)^{p^r} = X^{p^r} - a^{p^r}$$

  - if $a, c \in K$ satisfy $a^{p^r} = c$

$$X^{p^r} - c = X^{p^r} - a^{p^r} = (X-a)^{p^r}$$

    hence, polynomial $X^{p^r} - c$ has precisely one root a of multiplicity p^r!

Algebraic Extension

Algebraic extension

- will show
  - for polynomial over field, always exists some extension of *that* field where the polynomial has root
  - existence of algebraic closure for every field

Extension of field

Definition 86. [extension of field] *for field E and its subfield $F \subset E$, E said to be* extension field of F*, (sometimes) denoted by* E/F *(which should* not *confused with* factor group*)*

- *can view E as* vector space *over F*
- *if dimension of the vector space is finite, extension called* finite extension of F
- *if infinite, called* infinite extension of F

Algebraic over field

Definition 87. [algebraic over field] *for field E and its subfield $F \subset E$, $\alpha \in E$ satisfying*

$$(\exists a_0, \ldots, a_n \text{ with not all } a_i \text{ zero}) \, (a_0 + a_1\alpha + \cdots + a_n\alpha^n = 0)$$

said to be algebraic over F

- *for algebraic $\alpha \neq 0$, can always find such equation like above that $a_0 \neq 0$*

- equivalent statements to Definition 87
  - exists homeomorphism $\varphi : F[X] \to E$ such that

$$(\forall x \in F) \, (\varphi(x) = x) \;\; \& \;\; \varphi(X) = \alpha \;\; \& \;\; \mathrm{Ker}\, \varphi \neq \{0\}$$

  - exists evaluation homeomorphism $\mathrm{ev}_\alpha : F[X] \to E$ with nonzero kernel (refer to Definition 72 for definition of evaluation homeomorphism)

- in which case, $\operatorname{Ker}\varphi$ is principal ideal (by Theorem 14), hence generated by single element, thus exists nonzero $p(X) \in F[X]$ (with normalized leading coefficient being 1) so that
$$F[X]/(p(X)) \approx F[\alpha]$$

- $F[\alpha]$ *entire* (Lemma 6), hence $p(X)$ irreducible (refer to Definition 55)

Definition 88. [THE irreducible polynomial] *normalized $p(X)$ (i.e., with leading coefficient being 1) uniquely determined by α, called* THE irreducible polynomial of α over F, *denoted by* $\operatorname{Irr}(\alpha, F, X)$

Algebraic extensions

Definition 89. [algebraic extension] *for field F, its extension field every element of which is algebraic over F, said to be* algebraic extension of F

Proposition 22. [algebraicness of finite field extensions] *for field F, every finite extension field of F is algebraic over F*

- converse is *not* true, $e.g.$, subfield of complex numbers consisting of algebraic numbers over $\mathbf{Q}$ is infinite extension of $\mathbf{Q}$

Dimension of extensions

Definition 90. [dimension of extension] *for field F and its extension field E, dimension of E as vector space over F, called* dimension of E over F*, denoted by* $[E : F]$

Proposition 23. [dimension of finite extension] *for field k and its extension fields F and E with $k \subset F \subset E$*

$$[E : k] = [E : F][F : k]$$

- *if $\langle x_i \rangle_{i \in I}$ is basis for F over k, and $\langle y_j \rangle_{j \in J}$ is basis for E over F, $\langle x_i y_j \rangle_{(i,j) \in I \times J}$ is basis for E over k*

Corollary 10. [finite dimension of extension] *for field k and its extension fields F & E with $k \subset F \subset E$, E/k is finite* if and only if *both F/k and E/F are finite*

Generation of field extensions

Definition 91. [generation of field extensions] *for field k, its extension field E, and $\alpha_1, \ldots, \alpha_n \in E$, smallest subfield containing k and α_1, . . . , α_n, said to be* finitely generated over k by α_1, . . . , α_n, *denoted by* $k(\alpha_1, \ldots, \alpha_n)$

- $k(\alpha_1, \ldots, \alpha_n)$ consists of all quotients $f(\alpha_1, \ldots, \alpha_n)/g(\alpha_1, \ldots, \alpha_n)$ where $f, g \in k[X]$ and $g(\alpha_1, \ldots, \alpha_n) \neq 0$, $i.e.$

$$
\begin{aligned}
& k(\alpha_1, \ldots, \alpha_n) \\
& \quad = \quad \{f(\alpha_1, \ldots, \alpha_n)/g(\alpha_1, \ldots, \alpha_n) \,|\, f, g \in f[X], g(\alpha_1, \ldots, \alpha_n) \neq 0\}
\end{aligned}
$$

- *any* field extension E over k is union of smallest subfields containing $\alpha_1, \ldots, \alpha_n$ where $\alpha_1, \ldots, \alpha_n$ range over finite set of elements of E, $i.e.$

$$
E = \bigcup_{n \in \mathbf{N}} \bigcup_{\alpha_1, \ldots, \alpha_n \in E} k(\alpha_1, \ldots, \alpha_n)
$$

Proposition 24. [finite extension is finitely generated] *every finite extension of field is finitely generated*

Tower of fields

Definition 92. [tower of fields] *sequence of extension fields*

$$F_1 \subset F_2 \subset \cdots \subset F_n$$

called tower of fields

Definition 93. [finite tower of fields] *tower of fields, said to be* finite if and only if *each step of extensions is finite*

Algebraicness of finitely generated subfields

Proposition 25. [algebraicness of finitely generated subfield by single element] *for field k, its extension field E, and $\alpha \in E$ being algebraic over k*

$$k(\alpha) = k[\alpha]$$

and

$$[k(\alpha) : k] = \deg \mathrm{Irr}(\alpha, k, X)$$

hence $k(\alpha)$ is finite extension of k, thus algebraic extension over k *(by Proposition 22)*

Lemma 9. [a fortiori algebraicness] *for field k, its extension field F, and $\alpha \in E$ being algebraic over k where $k(\alpha)$ and F are subfields of common field, α is algebraic over F*

- *indeed,* $\mathrm{Irr}(\alpha, k, X)$ *has* a fortiori *coefficients in F*

- assume tower of fields

$$k \subset k(\alpha_1) \subset k(\alpha_1, \alpha_2) \subset \cdots \subset k(\alpha_1, \ldots, \alpha_n)$$

where α_i is algebraic over k

- then, α_{i+1} is algebraic over $k(\alpha_1, \ldots, \alpha_i)$ (by Lemma 9)

Proposition 26. [algebraicness of finitely generated subfields by multiple elements] *for field k and α_1, ..., α_n being algebraic over k, $E = k(\alpha_1, \ldots, \alpha_n)$ is finitely algebraic over k (due to Proposition 25, Proposition 23, and Proposition 22). Indeed, $E = k[\alpha_1, \ldots, \alpha_n]$ and*

$$\begin{aligned}[k(\alpha_1, \ldots, \alpha_n) : k] &= \deg \mathrm{Irr}(\alpha_1, k, X) \deg \mathrm{Irr}(\alpha_2, k(\alpha_1), X) \\ &\quad \cdots \deg \mathrm{Irr}(\alpha_n, k(\alpha_1, \ldots, \alpha_{n-1}), X),\end{aligned}$$

(proof can be found in Proof 9)

Compositum of subfields and lifting

Definition 94. [compositum of subfields] *for field k and its extension fields E and F, which are subfields of common field L, smallest subfield of L containing both E and F, called* compositum of E and F in L, *denoted by* EF

! cannot define compositum if E and F are not embedded in common field L

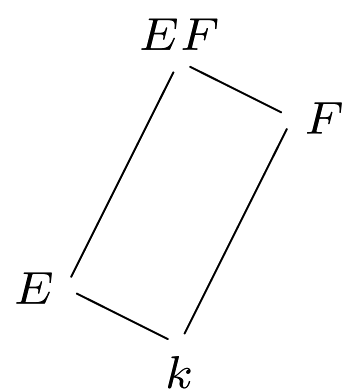
- could define *compositum of set of subfields of L* as smallest subfield containing subfields in the set

Lemma 10. *extension E of k is compositum of all its finitely generated subfields over k, i.e., $E = \bigcup_{n\in\mathbf{N}} \bigcup_{\alpha_1,\ldots,\alpha_n\in E} k(\alpha_1,\ldots,\alpha_n)$*

Lifting

Definition 95. [lifting] *extension EF of F, called* translation of E to F *or* lifting of E to F

- *often draw diagram as in the figure*



Finite generation of compositum

Lemma 11. [finite generation of compositum] *for field k, its extension field F, and $E = k(\alpha_1, \ldots, \alpha_n)$ where both E and F are contained in common field L,*

$$EF = F(\alpha_1, \ldots, \alpha_n)$$

$i.e.$, *compositum EF is finitely generated over F (proof can be found in Proof 10)*

- *refer to diagra in the figure*

$$\begin{tikzcd}
 & EF = F(\alpha_1, \ldots, \alpha_n) \arrow[dl, dash] \arrow[dr, dash] & \\
E = k(\alpha_1, \ldots, \alpha_n) \arrow[dr, dash] & & F \arrow[dl, dash] \\
 & k &
\end{tikzcd}$$

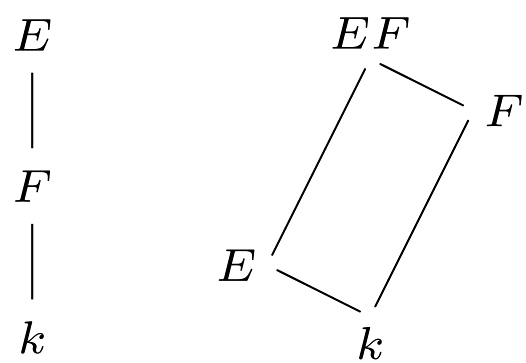
Distinguished classes

Definition 96. [distinguished class of field extensions] *for field k, class $\mathcal{C}$ of extension fields satisfying*

- *for tower of fields $k \subset F \subset E$, extension $k \subset E$ is in $\mathcal{C}$* if and only if *both $k \subset F$ and $F \subset E$ are in $\mathcal{C}$*
- *if $k \subset E$ is in $\mathcal{C}$, F is any extension of k, and both E and F are subfields of common field, then $F \subset EF$ is in $\mathcal{C}$*

said to be distinguished*; the figure illustrates these two properties, which imply the following property*

- *if $k \subset F$ and $k \subset E$ are in $\mathcal{C}$ and both E and F are subfields of common field, $k \subset EF$ is in $\mathcal{C}$*



Both algebraic and finite extensions are distinguished

Proposition 27. [algebraic and finite extensions are distinguished] *class of algebraic extensions is distinguished, so is class of finite extensions*

- true that finitely generated extensions form distinguished class (not necessarily algebraic extensions or finite extensions)

Field embedding and embedding extension

Definition 97. [field embedding] *for two fields F and L, injective homeomorphism $\sigma : F \to L$, called* embedding of F into L*; then (of course) σ induces isomorphism of F with its image σF*[1]

Definition 98. [field embedding extension] *for field embedding $\sigma : F \to L$, field extension $F \subset E$, and embedding $\tau : E \to L$ whose restriction to F being equal to σ, said to* be over σ *or* extend σ*; if σ is identity, embedding τ, called* embedding of E over F*; diagrams in the figure show these embedding extensions*

$$\begin{tikzcd} E \arrow[r, "\tau"] & L \\ F \arrow[u, "\mathrm{inc}"] \arrow[r, "\sigma"'] & L \arrow[u, "\mathrm{id}"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\tau"] & & A' \\ & A/\mathfrak{a} \arrow[ul, "\mathrm{inc}"] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

- assuming F, E, σ, and τ same as in Definition 98, if $\alpha \in E$ is root of $f \in F[X]$, then α^τ is root of f^σ for if $f(X) = \sum_{i=0}^n a_i X^i$, then $f(\alpha) = \sum_{i=0}^n a_i \alpha^i = 0$, and $0 = f(\alpha)^\tau = \sum_{i=0}^n (a_i^\tau)(\alpha^\tau)^i = \sum_{i=0}^n a_i^\sigma (\alpha^\tau)^i = f^\sigma(\alpha^\tau)$

[1] Here σF is sometimes written as F^σ.

Embedding of field extensions

Lemma 12. [field embedding of algebraic extension] *for field k and its algebraic extension E, embedding of E into itself over k is isomorphism*

Lemma 13. [compositums of fields] *for field k and its field extensions E and F contained in common field,*

$$E[F] = F[E] = \bigcup_{n=1}^{\infty} \{e_1 f_1 + \cdots + e_n f_n \,| e_i \in E, f_i \in F\}$$

and EF is field of quotients of these elements

Lemma 14. [embeddings of compositum of fields] *for field k, its field extensions E_1 and E_2 contained in commen field E, and embedding $\sigma : E \to L$ for field L,*

$$\sigma(E_1 E_2) = \sigma(E_1)\sigma(E_2)$$

Existence of roots of irreducible polynomial

- assume $p(X) \in k[X]$ irreducible polynomial and consider canonical map, which is ring homeomorphism
$$\sigma : k[X] \to k[X]/((p(X))$$
- consider $\mathrm{Ker}\,\sigma|k$
  - every kernel of ring homeomorphism is ideal, hence if nonzero $a \in \mathrm{Ker}\,\sigma|k$, $1 \in \mathrm{Ker}\,\sigma|k$ because $a^{-1} \in \mathrm{Ker}\,\sigma|k$, but $1 \not\in (p(X))$
  - thus, $\mathrm{Ker}\,\sigma|k = \{0\}$, hence $p^\sigma \neq 0$
- now for $\alpha = X^\sigma$
$$p^\sigma(\alpha) = p^\sigma(X^\sigma) = (p(X))^\sigma = 0$$
- thus, α is algebraic in k^σ, $i.e.$, $\alpha \in k[X]^\sigma$ is root of p^σ in $k^\sigma(\alpha)$

Lemma 15. [existence of roots of irreducible polynomial] *for field k and irreducible $p(X) \in k[X]$ with $\deg p \geq 1$, exist field L and homeomorphism $\sigma : k \to L$ such that p^σ with $\deg p^\sigma \geq 1$ has root in field extension of k^σ*

Existence of algebraically closed algebraic field extensions

Proposition 28. [existence of extension fields containing roots] *for field k and $f \in k[X]$ with $\deg f \geq 1$, exists extension of k in which f has root*

Corollary 11. [existence of extension fields containing roots] *for field k and f_1, $\ldots$, $f_n \in k[X]$ with $\deg f_i \geq 1$, exists extension of k in which every f_i has root*

Theorem 17. [existence of algebraically closed field extensions] *for every field k, exists algebraically closed extension of k*

Corollary 12. [existence of algebraically closed algebraic field extensions] *for every field k, exists algebraically closed algebraic extension of k (proof can be found in Proof 11)*

Isomorphism between algebraically closed algebraic extensions

Proposition 29. [number of algebraic embedding extensions] *for field, k, α being algebraic over k, algebraically closed field, L, and embedding, $\sigma : k \to L$, # possible embedding extensions of σ to $k(\alpha)$ in L is equal to # distinct roots of $\mathrm{Irr}(\alpha, k, X)$, hence no greater than # roots of $\mathrm{Irr}(\alpha, k, X)$*

Theorem 18. [algebraic embedding extensions] *for field, k, its algebraic extensions, E, algebraically closed field, L, and embedding, $\sigma : k \to L$, exists embedding extension of σ to E in L; if E is algebraically closed and L is algebraic over k^σ, every such embedding extension is isomorphism of E onto L*

Corollary 13. [isomorphism between algebraically closed algebraic extensions] *for field, k, and its algebraically closed algebraic extensions, E and E', exists isomorphism bewteen E and E' which induces identity on k, $i.e.$*

$$\tau : E \to E'$$

where $\tau|k$ is identity

- thus, *algebraically closed algebraic extension is determined up to isomorphism*

Algebraic closure

Definition 99. [algebraic closure] *for field, k, algebraically closed algebraic extension of k, which is determined up to isomorphism, called* algebraic closure of k, *frequently denoted by* k^{a}

- examples
  - complex conjugation is automorphism of **C** (which is the only continuous automorphism of **C**)
  - subfield of **C** consisting of all numbers which are algebraic over **Q** is algebraic closure of **Q**, $i.e.$, $\mathbf{Q}^{\mathrm{a}}$
  - $\mathbf{Q}^{\mathrm{a}} \neq \mathbf{C}$
  - $\mathbf{R}^{\mathrm{a}} = \mathbf{C}$
  - $\mathbf{Q}^{\mathrm{a}}$ *is countable*

Theorem 19. [countability of algebraic closure of finite fields] *algebraic closure of finite field is countable*

Theorem 20. [cardinality of algebraic extensions of infinite fields] *for infinite field, k, every algebraic extension of k has same cardinality as k*

Splitting fields

Definition 100. [splitting fields] *for field, k, and $f \in k[X]$ with $\deg f \geq 1$, field extension, K, of k, f splits into linear factors in which, $i.e.$,*

$$f(X) = c(X - \alpha_1) \cdots (X - \alpha_n)$$

and which is finitely generated over k by α_1, ..., α_n (hence $K = k(\alpha_1, \ldots, \alpha_n)$), called splitting field of f

- for field, k, every $f \in k[X]$ has splitting field in k^{a}

Theorem 21. [isomorphism between splitting fields] *for field, k, $f \in k[X]$ with $\deg f \geq 1$, and two splitting fields of f, K and E, exists isomorphism between K and E; if $k \subset K \subset k^{\mathrm{a}}$, every embedding of E into k^{a} over k is isomorphism of E onto K*

Splitting fields for family of polynomials

Definition 101. [splitting fields for family of polynomials] *for field, k, index set, Λ, and indexed family of polynomials, $\{f_\lambda \in k[X] | \lambda \in \Lambda, \deg f_\lambda \geq 1\}$, extension field of k, every f_λ splits into linear factors in which and which is generated by all roots of all polynomials, f_λ, called* splitting field for family of polynomials

- in most applications, deal with finite Λ
- becoming increasingly important to consider infinite algebraic extensions
- various proofs would not be simpler if restricted ourselves to finite cases

Corollary 14. [isomorphism between splitting fields for family of polynomials] *for field, k, index set, Λ, and two splitting fields, K and E, for family of polynomials, $\{f_\lambda \in k[X] | \lambda \in \Lambda, \deg f_\lambda \geq 1\}$, every embedding of E into K^{a} over k is isomorphism of E onto K*

Normal extensions

Theorem 22. [normal extensions] *for field, k, and its algebraic extension, K, with $k \subset K \subset k^{\mathrm{a}}$, following statements are equivalent*

- *every embedding of K into k^{a} over k induces automorphism*
- *K is splitting field of family of polynomials in $k[X]$*
- *every irreducible polynomial of $k[X]$ which has root in K splits into linear factors in K*

Definition 102. [normal extensions] *for field, k, and its algebraic extension, K, with $k \subset K \subset k^{\mathrm{a}}$, satisfying properties in Theorem 22, said to be normal*

- *not* true that class of normal extensions is *distinguished*
  - $e.g.$, below tower of fields is tower of normal extensions

$$\mathbf{Q} \subset \mathbf{Q}(\sqrt{2}) \subset \mathbf{Q}(\sqrt[4]{2})$$

  - but, extension $\mathbf{Q} \subset \mathbf{Q}(\sqrt[4]{2})$ is not normal because complex roots of $X^4 - 2$ are not in $\mathbf{Q}(\sqrt[4]{2})$

Retention of normality of extensions

Theorem 23. [retention of normality of extensions] *normal extensions remain normal under lifting; if $k \subset E \subset K$ and K is normal over k, K is normal over E; if K_1 and K_2 are normal over k and are contained in common field, $K_1 K_2$ is normal over k, and so is $K_1 \cap K_2$*

Separable degree of field extensions

- for field, F, and its algebraic extension, E
  - let L be algebraically closed field and assume embedding, $\sigma : F \to L$
    - exists embedding extension of σ to E in L by Theorem 18
    - such σ maps E on subfield of L which is algebraic over F^σ
    - hence, E^σ is contained in algebraic closure of F^σ which is contained in L
    - will *assume* that L is the algebraic closure of F^σ
  - let L' be another algebraically closed field and assume another embedding, $\tau : F \to L'$ - assume as before that L' is algebraic closure of F^τ
  - then Theorem 18 implies, exists isomorphism, $\lambda : L \to L'$ extending $\tau \circ \sigma^{-1}$ applied to F^σ
  - let S_σ & S_τ be sets of embedding extensions of σ to E in L and L' respectively
  - then λ induces map from S_σ into S_τ with $\tilde{\sigma} \mapsto \lambda \circ \tilde{\sigma}$ and λ^{-1} induces inverse map from S_τ into S_σ, hence exists bijection between S_σ and S_τ, hence have same cardinality

Definition 103. [separable degree of field extensions] *above cardinality only depends on extension E/F, called* separable degree of E over F, *denoted by* $[E : F]_s$

Multiplicativity of and upper bound on separable degree of field extensions

Theorem 24. [multiplicativity of separable degree of field extensions] *for tower of algebraic field extensions, $k \subset F \subset E$,*

$$[E : k]_s = [E : F]_s [F : k]_s$$

Theorem 25. [upper limit on separable degree of field extensions] *for finite algebraic field extension, $k \subset E$*

$$[E : k]_s \leq [E : k]$$

- $i.e.$, separable degree is at most equal to degree ($i.e.$, dimension) of field extension

Corollary 15. *for tower of algebraic field extensions, $k \subset F \subset E$, with $[E : k] < \infty$*

$$[E : k]_s = [E : k]$$

holds if and only if *corresponding equality holds in every step of tower, $i.e.$, for E/F and F/k*

Finite separable field extensions

Definition 104. [finite separable field extensions] *for finite algebraic field extension, E/k, with $[E:k]_s = [E:k]$, E, said to be* separable over k

Definition 105. [separable algebraic elements] *for field, k, α, which is algebraic over k with $k(\alpha)$ being separable over k, said to be* separable over k

Proposition 30. [separability and multiple roots] *for field, k, α, which is algebraic over k, is separable over k* if and only if *$\mathrm{Irr}(\alpha, k, X)$ has no multiple roots*

Definition 106. [separable polynomials] *for field, k, $f \in k[X]$ with no multiple roots, said to be* separable

Lemma 16. *for tower of algebraic field extensions, $k \subset F \subset K$, if $\alpha \in K$ is separable over k, then α is separable over F*

Theorem 26. [finite separable field extensions] *for finite field extension, E/k, E is separable over k* if and only if *every element of E is separable over k*

Arbitrary separable field extensions

Definition 107. [arbitrary separable field extensions] *for (not necessarily finite) field extension, E/k, E, of which every finitely generated subextension is separable over k, $i.e.$,*

$$(\forall n \in \mathbf{N} \;\&\; \alpha_1, \ldots, \alpha_n \in E)\,(k(\alpha_1, \ldots, \alpha_n) \text{ is separable over } k)$$

said to be separable over k

Theorem 27. [separable field extensions] *for algebraic extension, E/k, E, which is generated by family of elements, $\{\alpha_\lambda\}_{\lambda\in\Lambda}$, with every α_λ is separable over k, is separable over k*

Theorem 28. [separable extensions are distinguished] *separable extensions form distinguished class of extensions*

Separable closure and conjugates

Definition 108. [separable closure] *for field, k, compositum of all separable extensions of k in given algebraic closure k^{a}, called* separable closure of k*, denoted by* k^{s} or k^{sep}

Definition 109. [conjugates of fields] *for algebraic field extension, E/k, and embedding of E, σ, in k^{a} over k, E^{σ}, called* conjugate of E in k^{a}

- smallest normal extension of k containing E is compositum of all conjugates of E in E^{a}

Definition 110. [conjugates of elements of fields] *for field, k, α being algebraic over k, and distinct embeddings, $\sigma_1, \ldots, \sigma_r$ of $k(\alpha)$ into k^{a} over k, $\alpha^{\sigma_1}, \ldots, \alpha^{\sigma_r}$, called* conjugates of α in k^{a}

- $\alpha^{\sigma_1}, \ldots, \alpha^{\sigma_r}$ are simply distinct roots of $\mathrm{Irr}(\alpha, k, X)$
- smallest normal extension of k containing one of these conjugates is simply $k(\alpha^{\sigma_1}, \ldots, \alpha^{\sigma_r})$

Prime element theorem

Theorem 29. [prime element theorem] *for finite algebraic field extension, E/k, exists $\alpha \in E$ such that $E = k(\alpha)$* if and only if *exists only finite $\#$ fields, F, such that $k \subset F \subset E$; if E is separable over k, exists such element, α*

Definition 111. [primitive element of fields] *for finite algebraic field extension, E/k, $\alpha \in E$ with $E = k(\alpha)$, called* primitive element of E over k

Finite fields

Definition 112. [finite fields] *for every prime number, p, and integer, $n \geq 1$, exists finite field of order p^n, denoted by $\mathbf{F}_{p^n}$, uniquely determined as subfield of algebraic closure, $\mathbf{F}_p{}^{\mathrm{a}}$, which is splitting field of polynomial*

$$f_{p,n}(X) = X^{p^n} - X$$

and whose elements are roots of $f_{p,n}$

Theorem 30. [finite fields] *for every finite field, F, exist prime number, p, and integer, $n \geq 1$, such that $F = \mathbf{F}_{p^n}$*

Corollary 16. [finite field extensions] *for finite field, $\mathbf{F}_{p^n}$, and integer, $m \geq 1$, exists one and only one extension of degree, m, which is $\mathbf{F}_{p^{mn}}$*

Theorem 31. [multiplicative group of finite field] *multiplicative group of finite field is cyclic*

Automorphisms of finite fields

Definition 113. [Frobenius mapping] *mapping*

$$\varphi_{p,n} : \mathbf{F}_{p^n} \to \mathbf{F}_{p^n}$$

defined by $x \mapsto x^p$*, called* Frobenius mapping

- $\varphi_{p,n}$ is (ring) homeomorphism with $\operatorname{Ker} \varphi_{p,n} = \{0\}$ since $\mathbf{F}_{p^n}$ is field, thus is injective (Proposition 17), and surjective because $\mathbf{F}_{p^n}$ is finite,
- thus, $\varphi_{p,n}$ is *isomorphism leaving* $\mathbf{F}_p$ *fixed*

Theorem 32. [group of automorphisms of finite fields] *group of automorphisms of* $\mathbf{F}_{p^n}$ *is cyclic of degree* n*, generated by* $\varphi_{p,n}$

Theorem 33. [group of automorphisms of finite fields over another finite field] *for prime number,* p*, and integers,* $m, n \geq 1$*, in any* $\mathbf{F}_p{}^{\mathrm{a}}$*,* $\mathbf{F}_{p^n}$ *is contained in* $\mathbf{F}_{p^m}$ if and only if n *divides* m*,* $i.e.$*, exists* $d \in \mathbf{Z}$ *such that* $m = dn$*, in which case,* $\mathbf{F}_{p^m}$ *is* normal and separable *over* $\mathbf{F}_{p^n}$ *group of automorphisms of* $\mathbf{F}_{p^m}$ *over* $\mathbf{F}_{p^n}$ *is cyclic of order,* d*, generated by* $\varphi_{p,m}^n$

Galois Theory

What we will do to appreciate Galois theory

- study
  - group of automorphisms of finite (and infinite) Galois extension (at length)
  - give examples, *e.g.*, cyclotomic extensions, abelian extensions, (even) non-abelian ones
  - leading into study of matrix representation of Galois group & classifications

- have tools to prove
  - fundamental theorem of algebra
  - insolvability of quintic polynomials

- mention unsolved problems
  - given finite group, exists Galois extension of $\mathbf{Q}$ having this group as Galois group?

Fixed fields

Definition 114. [fixed field] *for field, K, and group of automorphisms, G, of K,*

$$\{x \in K | \forall \sigma \in G, x^\sigma = x\} \subset K$$

is subfield of K, and called fixed field of G, *denoted by* K^G

- K^G is subfield of K because for every $x, y \in K^G$
  - $0^\sigma = 0 \Rightarrow 0 \in K^G$
  - $(x+y)^\sigma = x^\sigma + y^\sigma = x + y \Rightarrow x + y \in K^G$
  - $(-x)^\sigma = -x^\sigma = -x \Rightarrow -x \in K^G$
  - $1^\sigma = 1 \Rightarrow 1 \in K^G$
  - $(xy)^\sigma = x^\sigma y^\sigma = xy \Rightarrow xy \in K^G$
  - $(x^{-1})^\sigma = (x^\sigma)^{-1} = x^{-1} \Rightarrow x^{-1} \in K^G$

  hence, K^G closed under addition & multiplication, and is commutative division ring, thus field
- $0, 1 \in K^G$, hence K^G *contains prime field*

Galois extensions and Galois groups

Definition 115. [Galois extensions] *algebraic extension, K, of field, k, which is normal and separable, said to be* Galois (extension of k) *or* Galois over k *considering K as embedded in k^{a}; for convenience, sometimes say* K/k is Galois

Definition 116. [Galois groups] *for field, k and its Galois extension, K, group of automorphisms of K over k, called* Galois group of K over k, *denoted by* $G(K/k)$, $G_{K/k}$, $\mathrm{Gal}(K/k)$, or (simply) G

Definition 117. [Galois group of polynomials] *for field, k, separable $f \in k[X]$ with $\deg f \geq 1$, and its splitting field, K/k, Galois group of K over k ($i.e.$, $G(K/k)$), called* Galois group of f over k

Proposition 31. [Galois group of polynomials and symmetric group] *for field, k, separable $f \in k[X]$ with $\deg f \geq 1$, and its splitting field, K/k,*

$$f(X) = (X - \alpha_1) \cdots (X - \alpha_n)$$

elements of Galois group of f over k, G, permute roots of f, hence, exists injective homeomorphism of G into S_n, $i.e.$, symmetric group on n elements

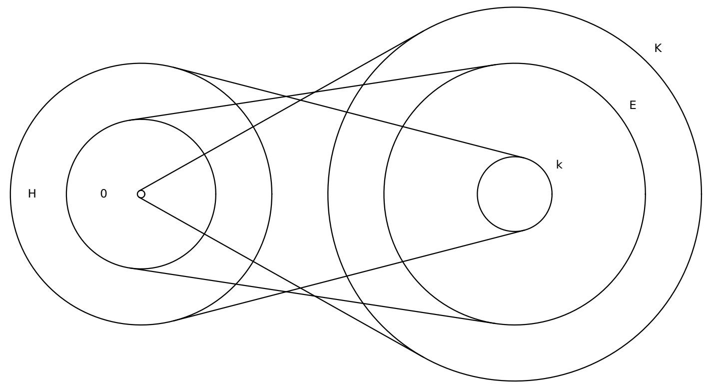
Fundamental theorem for Galois theory

Theorem 34. [fundamental theorem for Galois theory] *for finite Galois extension, K/k*

- *map $H \mapsto K^H$ induces isomorphism between set of subgroups of $G(K/k)$ & set of intermediate fields*
- *subgroup, H, of $G(K/k)$, is normal* if and only if *K^H/k is Galois*
- *for normal subgroup, H, $\sigma \mapsto \sigma|K^H$ induces isomorphism between $G(K/k)/H$ and $G(K^H/k)$*

(illustrated in the figure)

- shall prove step by step


H
0
K
E
k

Galois subgroups association with intermediate fields

Theorem 35. [Galois subgroups associated with intermediate fields - 1] *for Galois extension, K/k, and intermediate field, F*

- *K/F is Galois & $K^{G(K/F)} = F$, hence, $K^G = k$*
- *map*

$$F \mapsto G(K/F)$$

induces injective *homeomorphism from set of intermediate fields to subgroups of G*

(proof can be found in Proof 12)

Definition 118. [Galois subgroups associated with intermediate fields] *for Galois extension, K/k, and intermediate field, F, subgroup, $G(K/F) \subset G(K/k)$, called* group associated with F, *said to* belong to F

Corollary 17. [Galois subgroups associated with intermediate fields - 1] *for Galois extension, K/k, and two intermediate fields, F_1 and F_2, $G(K/F_1) \cap G(K/F_2)$ belongs to F_1F_2, $i.e.$,*

$$G(K/F_1) \cap G(K/F_2) = G(K/F_1F_2)$$

(proof can be found in Proof 13)

Corollary 18. [Galois subgroups associated with intermediate fields - 2] *for Galois extension, K/k, and two intermediate fields, F_1 and F_2, smallest subgroup of G containing $G(K/F_1)$ and $G(K/F_2)$ belongs to $F_1 \cap F_2$, $i.e.$*

$$\bigcap_{G(K/F_1)\subset H, G(K/F_2)\subset H} \{H | H \subset G(K/k)\} = G(K/(F_1 \cap F_2))$$

Corollary 19. [Galois subgroups associated with intermediate fields - 3] *for Galois extension, K/k, and two intermediate fields, F_1 and F_2,*

$$F_1 \subset F_2 \text{ if and only if } G(K/F_2) \subset G(K/F_1)$$

(proof can be found in *Proof 14**)*

Corollary 20. *for finite separable field extension, E/k, the smallest normal extension of k containing E, K, K/k is finite Galois and exist only finite number of intermediate fields*

Lemma 17. *for algebraic separable extension, E/k, if every element of E has degree no greater than n over k for some $n \geq 1$, E is finite over k and $[E : k] \leq n$*

Theorem 36. [Artin's theorem] *(Artin) for field, K, finite $\mathrm{Aut}(K)$ of order, n, and $k = K^{\mathrm{Aut}(K)}$, K/k is Galois, $G(K/k) = \mathrm{Aut}(K)$, and $[K : k] = n$*

Corollary 21. [Galois subgroups associated with intermediate fields - 4] *for finite Galois extension, K/k, every subgroup of $G(K/k)$ belongs to intermediate field*

Theorem 37. [Galois subgroups associated with intermediate fields - 2] *for Galois extension, K/k, and intermediate field, F,*

- *F/k is normal extension* if and only if *$G(K/F)$ is normal subgroup of $G(K/k)$*
- *if F/k is normal extension, map, $\sigma \mapsto \sigma|F$, induces homeomorphism of $G(K/k)$ onto $G(F/k)$ of which $G(K/F)$ is kernel, thus*

$$G(F/k) \approx G(K/k)/G(K/F)$$

Proof for fundamental theorem for Galois theory

- finally, we prove *fundamental theorem for Galois theory* (Theorem 34)
- assume K/k is finite Galois extension and H is subgroup of $G(K/k)$
  - Corollary 21 implies K^H is intermediate field, hence Theorem 35 implies K/K^H is Galois, Theorem 36 implies $G(K/K^H) = H$, thus, every H is Galois
  - map, $H \mapsto K^H$, induces homeomorphism, σ, of set of all subgroups of $G(K/k)$ into set of intermediate fields
  - σ is *injective* since for any two subgroups, H and H', of $G(K/k)$, if $K^H = K^{H'}$, then $H = G(K/K^H) = G(K/K^{H'}) = H'$
  - σ is *surjective* since for every intermediate field, F, Theorem 35 implies K/F is Galois, $G(K/F)$ is subgroup of $G(K/k)$, and $K^{G(K/F)} = F$, thus, $\sigma(G(K/F)) = K^{G(K/F)} = F$
  - *therefore, σ is isomorphism between set of all subgroups of $G(K/k)$ and set of intermediate fields*
  - since Theorem 28 implies separable extensions are distinguished, H^K/k is separable, thus Theorem 37 implies that K^H/k is Galois *if and only if* $G(K/K^H)$ is normal
  - lastly, Theorem 37 implies that if K^H/k is Galois, $G(H^K/k) \approx G(K/k)/H$

Abelian and cyclic Galois extensions and groups

Definition 119. [abelian Galois extensions] *Galois extension with abelian Galois group, said to be* abelian

Definition 120. [cyclic Galois extensions] *Galois extension with cyclic Galois group, said to be* cyclic

Corollary 22. *for Galois extension, K/k, and intermediate field, F,*

- *if K/k is abelian, F/k is Galois and abelian*
- *if K/k is cyclic, F/k is Galois and cyclic*

Definition 121. [maximum abelian extension] *for field, k, compositum of all abelian Galois extensions of k in given k^{a}, called* maximum abelian extension of k, *denoted by* k^{ab}

Theorems and corollaries about Galois extensions

Theorem 38. *for Galois extension, K/k, and arbitrary extension, F/k, where K and F are subfields of common field,*

- *KF/F and $K/(K \cap F)$ are Galois extensions*
- *map*

$$\sigma \mapsto \sigma|K$$

induces isomorphism between $G(KF/F)$ and $G(K/(K \cap F))$

theorem illustrated in the figure

$$\begin{tikzcd}
 & KF \arrow[dl, dash] \arrow[dr, dash, "\text{Galois}"] & \\
K \arrow[dr, dash, "\text{Galois}"'] & & F \arrow[dl, dash] \\
 & K \cap F \arrow[d, dash] & \\
 & k &
\end{tikzcd}$$

Corollary 23. *for finite Galois extension, K/k, and arbitrary extension, F/k, where K and F are subfields of common field,*

$$[KF : F] \text{ divides } [F : k]$$

Theorem 39. *for Galois extensions, K_1/k and K_2/k, where K_1 and K_2 are subfields of common field,*

- *K_1K_2/k is Galois extension*
- *map*

  $$\sigma \mapsto (\sigma|K_1, \sigma|K_2)$$

  of $G(K_1K_2/k)$ into $G(K_1/k) \times G(K_2/k)$ is injective; if $K_1 \cap K_2 = k$, map is isomorphism

theorem illustrated in the figure

$$
\begin{tikzcd}[column sep=small]
 & K_1K_2 \arrow[dl, dash] \arrow[dr, dash] & \\
K_1 \arrow[dr, dash] & & K_2 \arrow[dl, dash] \\
 & K_1 \cap K_2 \arrow[d, dash] & \\
 & k &
\end{tikzcd}
$$

Corollary 24. *for n Galois extensions, K_i/k, where K_1, ..., K_n are subfields of common field and $K_{i+1} \cap (K_1 \cdots K_i) = k$ for $i = 1, \ldots, n-1$,*

- *$K_1 \cdots K_n/k$ is Galois extension*
- *map*
$$\sigma \mapsto (\sigma|K_1, \ldots, \sigma|K_n)$$
induces isomorphism of $G(K_1 \cdots K_n/k)$ onto $G(K_1/k) \times \cdots \times G(K_n/k)$

Corollary 25. *for Galois extension, K/k, where $G(K/k)$ can be written as $G_1 \times \cdots \times G_n$, and K_1, . . . , K_n, each of which is fixed field of*

$$G_1 \times \cdots \times \underbrace{\{e\}}_{i\text{th position}} \times \cdots \times G_n$$

- K_1/k, . . . , K_n/k *are Galois extensions*
- $G(K_i/k) = G_i$ *for* $i = 1, \ldots, n$
- $K_{i+1} \cap (K_1 \cdots K_i) = k$ *for* $i = 1, \ldots, n-1$
- $K = K_1 \cdots K_n$

Theorem 40. *assume all fields are subfields of common field*

- *for two abelian Galois extensions, K/k and L/k, KL/k is abelian Galois extension*
- *for abelian Galois extension, K/k, and any extension, E/k, KE/E is abelian Galois extension*
- *for abelian Galois extension, K/k, and intermediate field, E, both K/E and E/k are abelian Galois extensions*

Solvable and radical extensions

Definition 122. [sovable extensions] *finite separable extension, E/k, such that Galois group of smallest Galois extension, K/k, containing E is solvable, said to be* solvable

Theorem 41. [solvable extensions are distinguished] *solvable extensions form distinguished class of extensions*

Definition 123. [solvable by radicals] *finite extension, F/k, such that it is separable and exists finite extension, E/k, containing F admitting tower decomposition*

$$k = E_0 \subset E_1 \subset \cdots \subset E_m = E$$

with E_{i+1}/E_i is obtained by adjoining root of

- *unity, or*
- *$X^n = a$ with $a \in E_i$, and n prime to characteristic, or*
- *$X_p - X - a$ with $a \in E_i$ if p is positive characteristic*

said to be solvable by radicals

Theorem 42. [extensions solvable by radicals] *separable extension, E/k, is solvable by radicals* if and only if *it is solvable*

Applications of Galois theory

Theorem 43. [insolvability of quintic polynomials] *general equation of degree, n, cannot be solved by radicals for $n \geq 5$ (implied by Definition 117, Proposition 31, Theorem 42, and Theorem 5)*

Theorem 44. [fundamental theorem of algebra] *$f \in \mathbf{C}[X]$ of degree, n, has precisely n roots in $\mathbf{C}$ (when counted with multiplicity), hence $\mathbf{C}$ is algebraically closed*

Selected Proofs

Selected proofs

- **Proof 1.** *(Proof for "relation among coset indices" on page 20)*

  Let $\{h_1, \ldots, h_n\}$ and $\{k_1, \ldots, k_m\}$ be coset representations of H in G and K in H respectively. Then $n = (G : H)$ and $m = (H : K)$. Note that $\bigcup_{i,j} h_i k_j K = \bigcup_i h_i H = G$, and if $h_i k_j K = h_k k_l K$ for some $1 \leq i, k \leq n$ and $1 \leq j, k \leq m$, $h_i k_j K H = h_k k_l K H \Leftrightarrow h_i k_j H = h_k k_l H \Leftrightarrow h_i H = h_j H \Leftrightarrow h_i = h_j$, thus $k_j K = k_l K$, hence $k_j = k_l$. Thus $\{h_i k_j | 1 \leq i \leq n, 1 \leq j \leq m\}$ is cosets representations of K in G, therefore $(G : K) = mn = (G : H)(H : K)$. ∎

- **Proof 2.** *(Proof for "normality and commutativity of commutator subgroups" on page 34)*

  – For $a, x, y \in G$,

$$
\begin{aligned}
axyx^{-1}y^{-1} &= ax(a^{-1}x^{-1}xa)yx^{-1}y^{-1}(a^{-1}a) \\
&= (axa^{-1}x^{-1})(x(ay)x^{-1}(ay)^{-1})a
\end{aligned}
$$

and

$$\begin{aligned} xyx^{-1}y^{-1}a &= (aa^{-1})xyx^{-1}(ay^{-1}ya^{-1})y^{-1}a \\ &= a((a^{-1}x)y(a^{-1}x)^{-1}y^{-1})(ya^{-1}y^{-1}a), \end{aligned}$$

hence commutator subgroup of G propagate every element of G from fron to back and vice versa. Therefore for every $a \in G$, $aG^C = G^C a$.

- For $x, y \in G$, $xG^C yG^C = xyG^C = G^C xy = (G^C x)(G^C y)$, hence G/G^C is commutative.
- For a homeomorphism of G, f, into a commutative group, and $x, y \in G$,

$$f(xyx^{-1}y^{-1}) = f(x)f(y)f(x^{-1})f(y^{-1}) = f(x)f(x^{-1})f(y)f(y^{-1}) = e$$

thus $xyx^{-1}y^{-1} \in \mathrm{Ker}\, f$, hence $G^C \subset \mathrm{Ker}\, f$.

∎

- **Proof 3.** *(Proof for "set of functions into ring is ring" on page 56)*
  - First, we show that the mapping addition defines a commutative additive group in $\mathrm{Map}(S, A)$. The addition is associative because A is a ring, hence defines an

additive (abelian) group, thus, monoids (Definition 4 & Definition 5), $i.e.$,

$$
\begin{aligned}
&(\forall f, g, h \in \mathrm{Map}(S, A)) \\
&\quad (\forall x \in S)\,(\quad ((f+g)+h)(x) = (f(x)+g(x))+h(x) \\
&\qquad\qquad\qquad = f(x)+(g(x)+h(x)) = (f+(g+h))(x)) \\
&\qquad\qquad \Rightarrow \qquad (f+g)+h = f+(g+h).
\end{aligned}
$$

Thus, the mapping addition defines an additive monoid in $\mathrm{Map}(S, A)$ with the zero mapping whose value is the additive unit element of A as the additive unit element of $\mathrm{Map}(S, A)$ (Definition 4). Now for every $f \in R$, a mapping $g \in R$ defined by $x \mapsto -f(x)$ satisfies $f+g = g+f = 0$, hence is the inverse of f. Therefore the additive monoid is a group (Definition 5). We further note that the addition is commutative because the additive group of A is abelian (Definition 36), $i.e.$,

$$
\begin{aligned}
&(\forall f, g \in S) \\
&\quad (\forall x \in M)\,(\quad (g+f)(x) = g(x)+f(x) = f(x)+g(x) = (f+g)(x)) \\
&\qquad \Rightarrow \qquad f+g = g+f.
\end{aligned}
$$

Therefore, the mapping addition defines a commutative additive group in $\mathrm{End}(M)$.

– The mapping multiplication is associative because A is ring, hence defines a multiplicative monoid, *i.e.*,

$$
\begin{aligned}
&(\forall f, g, h \in \mathrm{Map}(S, A)) \\
&\quad (\forall x \in S) \left(\quad
\begin{aligned}
&((fg)h)(x) = (fg)(x)h(x) = (f(x)g(x))h(x) \\
&= f(x)(g(x)h(x)) = f(x)(gh)(x) = (f(gh))(x) \right)
\end{aligned} \\
&\qquad\quad \Rightarrow \qquad (fg)h = f(gh).
\end{aligned}
$$

Thus, the mapping multiplication defines a multiplicative monoid in $\mathrm{Map}(S, A)$ with the mapping whose value is the multiplicative unit element of A as the multiplicative unit element (Definition 4).

– Now we show that the multiplication is distributive over addition in $\mathrm{Map}(S, A)$. Similary this is due to that the multiplication is distributive over addition in A. Note that

$$
\begin{aligned}
&(\forall f, g, h \in \mathrm{Map}(S, A)) \\
&\quad (\forall x \in S)\ (\quad (f(g+h))(x) = f(x)(g+h)(x) = f(x)(g(x)+h(x)) \\
&\qquad\qquad\qquad\quad = f(x)g(x) + f(x)h(x) = (fg)(x) + (fh)(x)) \\
&\qquad\quad \Rightarrow \qquad f(g+h) = fg + fh.
\end{aligned}
$$

We can similarly show that

$$
(\forall f, g, h \in \mathrm{Map}(S, A))\ ((f+g)h = fh + gh)\ .
$$

Therefore $\mathrm{Map}(S, A)$ is is ring (Definition 36). ∎

- **Proof 4.** *(Proof for "set of group endomorphisms is ring" on page 56)*
  - First, we show that the addition defines a commutative additive group in $\mathrm{End}(M)$. The addition is associative because M is group, hence, monoids (Definition 4 &

Definition 5), *i.e.*,

$$
\begin{aligned}
&(\forall f, g, h \in \operatorname{End}(M)) \\
&\quad (\forall x \in M)\,(\quad ((f+g)+h)(x) = (f(x)+g(x))+h(x) \\
&\qquad\qquad\qquad\qquad = f(x)+(g(x)+h(x)) = (f+(g+h))(x)) \\
&\qquad\qquad \Rightarrow \qquad\quad (f+g)+h = f+(g+h).
\end{aligned}
$$

Thus, the addition defines an additive monoid in $\operatorname{End}(M)$ with the zero mapping whose values is the unit element of M as the additive unit element (Definition 4). Now for every $f \in \operatorname{End}(M)$, a mapping $g \in \operatorname{End}(M)$ defined by $x \mapsto -f(x)$ satisfies $f+g = g+f = 0$, hence is the inverse of f. Therefore the addition defines the additive group in $\operatorname{End}(M)$ (Definition 5). We further note that the addition is commutative because M is abelian, *i.e.*,

$$
\begin{aligned}
&(\forall f, g \in \operatorname{End}(M))\,(\forall x \in M) \\
&\qquad ((g+f)(x) = g(x)+f(x) = f(x)+g(x) = (f+g)(x))\,.
\end{aligned}
$$

Therefore, the addition defines a commutative additive group in $\mathrm{End}(M)$.

– The multiplication is associative because the mapping composition is an associative operation, *i.e.*, $(\forall f, g, h \in \mathrm{End}(M))\left((f \circ g) \circ h = f \circ (g \circ h)\right)$, hence, the mapping composition defines a multiplicative monoid in $\mathrm{End}(M)$ with the identity mapping as the multiplicative unit element (Definition 4).

– Now we show that the multiplication is distributive over addition. Note that

$$
\begin{aligned}
&(\forall f, g, h \in \mathrm{End}(M)) \\
&\quad (\forall x \in M) \left(\quad (f \circ (g + h))(x) = f(g(x) + h(x)) \right. \\
&\qquad\qquad\qquad\qquad \left. = (f \circ g)(x) + (f \circ h)(x) \right) \\
&\qquad\qquad \Rightarrow \qquad f \circ (g + h) = (f \circ g) + (f \circ h).
\end{aligned}
$$

We can similarly show that

$$
(\forall f, g, h \in \mathrm{End}(M))\left((f + g) \circ h = (f \circ h) + (g \circ h)\right).
$$

Therefore for abelian group M, *set $\mathrm{End}(M)$ of group homeomorphisms of M into itself* is ring (Definition 36). ∎

- **Proof 5.** *(Proof for "nonzero ideals of integers are principal" on page 62)*

  Suppose $\mathfrak{a}$ is a nonzero ideal of **Z**. Because if negative integer, n, is in $\mathfrak{a}$, $-n$ is also in $\mathfrak{a}$ because $\mathfrak{a}$ is an additive group in the ring, **Z**. Thus, $\mathfrak{a}$ has at least one positive integer. By Principle **??**, there exists the smallest positive integer in $\mathfrak{a}$. Let n be that integer. Let $m \in \mathfrak{a}$. By Theorem 13, there exist $q, r \in$ **Z** such that $m = qn + r$ with $0 \leq r < n$. Since by the definition of ideals of rings (Definition 45) $\mathfrak{a}$ is an additive group in **Z**, hence $m - qn = r$ is also in $\mathfrak{a}$, thus r should be 0 because we assume n is the smallest positive integer in $\mathfrak{a}$. Thus $\mathfrak{a} = \{qn | q \in \mathbf{Z}\} = n\mathbf{Z}$. Therefore the ideal is either $\{0\}$ or $n\mathbf{Z}$ for some $n > 0$. Both $\{0\}$ and $n\mathbf{Z}$ are ideal. ∎

- **Proof 6.** *(Proof for "ideal generated by elements of ring" on page 64)*

  For all $x \in (a_1, \ldots, a_n)$, and $y \in A$ $yx = y\left(\sum x_i a_i\right) = \sum (y x_i) a_i$ for some $\langle x_i \rangle_{i=1}^n \subset A$, hence $yx \in A$, and $(a_1, \ldots, a_n)$ is additive group, thus is ideal of A, hence

  $$\bigcap_{\mathfrak{a}:\text{ideal containing } a_1,\ldots,a_n} \mathfrak{a} \subset (a_1, \ldots, a_n)$$

  Conversely, if $\mathfrak{a}$ contains $a_1, \ldots, a_n$, $Aa_i \subset \mathfrak{a}$, hence for every sequence, $\langle x_i \rangle_{i=1}^n \subset A$, $\sum x_i a_i \subset \mathfrak{a}$ because $\mathfrak{a}$ is additive subgroup of A, thus $(a_1, \ldots, a_n)$ is contained in every ideal containing $a_1, \ . \ . \ . \ , a_n$, hence

$$(a_1, \ldots, a_n) \subset \bigcap_{\mathfrak{a}\text{:ideal containing } a_1,\ldots,a_n} \mathfrak{a}$$

∎

- **Proof 7.** *(Proof for "kernel of ring-homeomorphism is ideal" on page 66)*
  Let $\mathrm{Ker}\, f$ be the kernel of a ring homeomorphism $f : A \to B$. Then Definition 52 implies

$$(\forall a, b \in \mathrm{Ker}\, f)\, (f(a+b) = f(a) + f(b) = 0 + 0 = 0 \Rightarrow a + b \in \mathrm{Ker}\, f)$$

hence, $\mathrm{Ker}\, f$ is closed under addition. Also Definition 52 implies

$$(\forall a \in \mathrm{Ker}\, f)$$
$$(f(-a) = f((-1)a) = f(-1)f(a) = f(-1)0 = 0 \Rightarrow -a \in \mathrm{Ker}\, f)$$

hence, every element of $\mathrm{Ker}\, f$ has its inverse. Also $0 \in \mathrm{Ker}\, f$ because $f(0) = 0$ by Definition 52. Thus, $\mathrm{Ker}\, f$ is a subgroup of A as additive group. Definition 52 also implies

$$(\forall a \in A, x \in \mathrm{Ker}\, f)$$
$$(f(ax) = f(a)f(x) = f(a)0 = 0 \;\&\; f(xa) = f(x)f(a) = 0f(a) = 0)$$

hence, $\mathrm{Ker}\, f$ is a two-side ideal, $i.e.$, an ideal. ∎

- **Proof 8.** *(Proof for "image of ring-homeomorphism is subring" on page 70)*
  Let $f : A \to B$ be a ring-homeomorphism for two rings A and B.
  - Then for any $z, w \in f(A)$, there exist $x, y \in A$ such that $f(x) = z$ and $f(y) = w$, hence Definition 52 implies

    $$z + w = f(x) + f(y) = f(x + y) \in f(A)$$

    because $x + y \in A$, hence $f(A)$ is closed under addition. Because $0 \in A$, Definition 52 implies $0 = f(0) \in f(A)$, hence $f(A)$ contains the additive unit element. Also, for every $z \in f(A)$, there exist $x \in A$ such that $f(x) = z$, but there exists $-x \in A$ because a ring is a commutative group with respect to addition

(Definition 36) thus, $f(-x) \in f(A)$, hence Definition 52 implies

$$f(-x) + z = f(-x) + f(x) = f(-x + x) = f(0) = 0$$

and the additive inverse of z, which is $f(-x)$, is in $f(A)$. Therefore $f(A)$ is an additive group. Lastly for any $z, w \in f(A)$, there exist $x, y \in A$ such that $f(x) = z$ and $f(y) = w$, hence Definition 36 implies

$$z + w = f(x) + f(y) = f(x + y) = f(y + x) = f(y) + f(x) = w + z,$$

thus,

$$f(A) \subset B \text{ is a commutative group with respect to addition.} \tag{1}$$

– Then for any $z, w \in f(A)$, there exist $x, y \in A$ such that $f(x) = z$ and $f(y) = w$, hence Definition 52 implies

$$zw = f(x)f(y) = f(xy) \in f(A)$$

because $xy \in A$, hence $f(A)$ is closed under multiplication. Because $1 \in A$, Definition 52 implies $1 = f(1) \in f(A)$, hence $f(A)$ contains the multiplicative unit element, thus,

$$f(A) \subset B \text{ is a monoid with respect to multiplication.} \tag{2}$$

Therefore $f(A) \subset B$ is a subring of B by (1) and (2). ∎

- **Proof 9.** *(Proof for "algebraicness of smallest subfields" on page 106)*
  Proposition 25 implies that $k(\alpha_1) = k[\alpha_1]$ and $[k(\alpha_1) : k] = \deg \mathrm{Irr}(\alpha_1, k, X)$. Because α_2 is algebraic over k, hence algebraic over $k(\alpha_1)$ *a fortiori*, thus, the same proposition implies

$$k(\alpha_1, \alpha_2) = (k(\alpha_1))[\alpha_2] = (k[\alpha_1])[\alpha_2] = k[\alpha_1, \alpha_2]$$

and

$$[k(\alpha_1, \alpha_2) : k(\alpha_1)] = \deg \mathrm{Irr}(\alpha_2, k(\alpha_1), X)$$

hence Proposition 23 implies

$$\begin{aligned} [k(\alpha_1, \alpha_2) : k] &= [k(\alpha_1, \alpha_2) : k(\alpha_1)][k(\alpha_1) : k] \\ &= \deg \mathrm{Irr}(\alpha_1, k, X) \deg \mathrm{Irr}(\alpha_2, k(\alpha_1), X). \end{aligned}$$

Using the mathematical induction, it is straightforward to show that

$$k(\alpha_1, \ldots, \alpha_n) = k[\alpha_1, \ldots, \alpha_n]$$

and

$$\begin{aligned}[k(\alpha_1, \ldots, \alpha_n) : k] \quad &= \quad \deg \mathrm{Irr}(\alpha_1, k, X) \deg \mathrm{Irr}(\alpha_2, k(\alpha_1), X) \\ &\quad\quad \cdots \deg \mathrm{Irr}(\alpha_n, k(\alpha_1, \ldots, \alpha_{n-1}), X),\end{aligned}$$

thus Proposition 22 implies that $k(\alpha_1, \ldots, \alpha_n)$ is finitely algebraic over k. ▮

- **Proof 10.** *(Proof for "finite generation of compositum" on page 109)*
  First, it is obvious that $E = k(\alpha_1, \ldots, \alpha_n) \subset F(\alpha_1, \ldots, \alpha_n)$ and $F \subset F(\alpha_1, \ldots, \alpha_n)$, hence $EF \subset F(\alpha_1, \ldots, \alpha_n)$ because EF is defined to be the smallest subfield that contains both E and F. Now every subfield containing both E and F contains all $f(\alpha_1, \ldots, \alpha_n)$ where $f \in F[X]$, hence all $f(\alpha_1, \ldots, \alpha_n)/g(\alpha_1, \ldots, \alpha_n)$ where $f, g \in F[X]$ and $g(\alpha_1, \ldots, \alpha_n) \neq 0$. Thus, $F(\alpha_1, \ldots, \alpha_n) \subset EF$ again by definition. Therefore $EF = F(\alpha_1, \ldots, \alpha_n)$. ▮

- **Proof 11.** *(Proof for "existence of algebraically closed algebraic extensions" on page 115)*

  Theorem 17 implies there exists an algebraically closed extension of k. Let E be such one. Let K be union of all algebraic extensions of k contained in E, then K is algebraic over k. Since k is algebraic over itself, K is not empty. Let $f \in K[X]$ with $\deg f \geq 1$. If α is a root of f, $\alpha \in E$. Since $K(\alpha)$ is algebraic over K and K is algebraic over k, $K(\alpha)$ is algebraic over k by Proposition 27. Therefore $K(\alpha) \subset K$ and $\alpha \in K$. Thus, K is algebraically closed algebraic extension of k. ∎

- **Proof 12.** *(Proof for "theorem - Galois subgroups associated with intermediate fields" on page 136)*

  Suppsoe $\alpha \in K^G$ and let $\sigma : k(\alpha) \to K^{\mathrm{a}}$ be an embedding inducing the identity on k. If we let $\tau : K \to K^{\mathrm{a}}$ extend σ, τ is automorphism by normality of K/k (Definition 102), hence $\tau \in G$, thus τ fixed α, which means σ is the identity, which is the only embedding extension of the identity embedding of k onto itself to $k(\alpha)$, thus, by Definition 103,

  $$[k(\alpha) : k]_s = 1.$$

  Since K is separable over k, α is separable over k (by Theorem 26), and $k(\alpha)$ is separable over k (by Definition 105), thus $[k(\alpha) : k] = [k(\alpha) : k]_s = 1$, hence $k(\alpha) = k$, thus $\alpha \in k$, hence

$$K^G \subset k.$$

Since by definition, $k \subset K^G$, we have $K^G = k$.

Now since K/k is a normal extension, K/F is also a normal extension (by Theorem 23). Also, since K/k is a separable extension, K/F is also separable extension (by Theorem 28 and Definition 96). Thus, K/F is Galois (by Definition 115).

Now let F and F' be two intermediate fields. Since $K^{G(K/k)} = k$, we have $K^{G(K/F)} = F$ and $K^{G(K/F')} = F'$, thus if $G(K/F) = G(K/F')$, $F = F'$, hence the map is injective. ∎

- **Proof 13.** *(Proof for "Galois subgroups associated with intermediate fields - 1" on page 136)*

  First, K/F_1 and K/F_2 are Galois extensions by Theorem 35, hence $G(K/F_1)$ and $G(K/F_2)$ can be defined. Also, Theorem 23 and Theorem 28 imply that K/F_1F_2 is Galois extension, hence $G(K/F_1F_2)$ can be defined, too.

  Every automorphism of G leaving both F_1 and F_2 leaves F_1F_2 fixed, hence $G(K/F_1) \cap G(K/F_2) \subset G(K/F_1F_2)$. Conversely, every automorphism of G leaving

F_1F_2 fxied leaves both F_1 and F_2 fixed, hence $G(K/F_1F_2) \subset G(K/F_1) \cap G(K/F_2)$. Now we can do the same thing using rather mathematically rigorous terms. Assume that $\sigma \in G(K/F_1) \cap G(K/F_2)$. Then

$$(\forall x \in F_1, y \in F_2) \left(x^\sigma = x \;\&\; y^\sigma = y\right),$$

thus

$$\begin{aligned}
&(\forall n, m \in \mathbf{N}) \\
&\quad \left(\forall x_1, \ldots, x_n, x_1', \ldots, x_m' \in F_1, y_1, \ldots, y_n, y_1', \ldots, y_m' \in F_2\right) \\
&\quad \left(\left(\frac{x_1y_1 + \cdots + x_ny_n}{x_1'y_1' + \cdots + x_m'y_m'}\right)^\sigma = \frac{x_1y_1 + \cdots + x_ny_n}{x_1'y_1' + \cdots + x_m'y_m'}\right),
\end{aligned}$$

hence $\sigma \in G(K/F_1F_2)$, thus $G(K/F_1) \cap G(K/F_2) \subset\in G(K/F_1F_2)$. Conversely if $\sigma \in G(K/F_1F_2)$,

$$(\forall x \in F_1, y \in F_2) \left(x^\sigma = x \;\&\; y^\sigma = y\right),$$

hence $\sigma \in G(K/F_1) \cap G(K/F_2)$, thus $G(K/F_1) \cap G(K/F_2) \subset G(K/F_1F_2)$. ∎

- **Proof 14.** *(Proof for "Galois subgroups associated with intermediate fields - 3" on page 137)*

  First, K/F_1 and K/F_2 are Galois extensions by Theorem 35, hence $G(K/F_1)$ and $G(K/F_2)$ can be defined.

  If $F_1 \subset F_2$, every automorphism leaving F_2 fixed leaves F_1 fixed, hence it is in $G(K/F_1)$, thus $G(K/F_2) \subset G(K/F_1)$. Conversely, if $G(K/F_2) \subset G(K/F_1)$, every intermediate field $G(K/F_1)$ leaves fixed is left fixed by $G(K/F_2)$, hence $F_1 \subset F_2$.

  Now we can do the same thing using rather mathematically rigorous terms. Assume $F_1 \subset F_2$ and that $\sigma \subset G(K/F_2)$. Since Theorem 35 implies that

$$F_1 \subset F_2 = \{x \in K | (\forall \sigma \in G(K/F_2))(x^\sigma = x)\},$$

  hence $(\forall x \in F_1)\,(x^\sigma = x)$, thus $\sigma \in G(K/F_1)$, hence

$$G(K/F_2) \subset G(K/F_1).$$

Conversely, assume that $G(K/F_2) \subset G(K/F_1)$. Then

$$\begin{aligned} F_1 &= \{x \in K | (\forall \sigma \in G(K/F_1))(x^\sigma = x)\} \\ &\subset \quad \{x \in K | (\forall \sigma \in G(K/F_2))(x^\sigma = x)\} = F_2 \end{aligned}$$

$\blacksquare$

References

References

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